p-adic valuation and linear forms in Fibonacci-like recurrence sequences

Maribel del Carmen Díaz Noguera

Universidad del Cauca
Facultad de Ciencias Naturales, Exactas y de la Educación
Departamento de Matemáticas
Doctorado en Ciencias Matemáticas
Popayán
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Director
Dr. Jhon Jairo Bravo Grijalba
Profesor de la Universidad del Cauca

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Resumen

Para un entero $k \geq 2$, consideremos la sucesión $k-$Fibonacci, la cual comienza con $0, \ldots, 0, 1$ (k términos) y a partir de ahí cada término de la sucesión es la suma de los k precedentes. Con $k = 2$ obtenemos los números de Fibonacci, mientras que si $k = 3$, la sucesión resultante es la sucesión Tribonacci, y así sucesivamente. La sucesión de Pell $\{P_n\}_{n\geq 0}$ es la sucesión lineal de orden dos definida por $P_n = 2P_{n-1} + P_{n-2}$ para $n \geq 2$, con condiciones iniciales $P_0 = 0$ y $P_1 = 1$. En esta tesis estamos interesados en encontrar todos los enteros c que tengan al menos dos representaciones como una diferencia entre un número $k-$Fibonacci y un número de Pell. Este problema, que puede verse como una variación del ya conocido problema de Pillai, generaliza trabajos previos concernientes a los casos $c = 0$ y $k = 2$. También encontramos todas las razones de sumas de dos números de Fibonacci que son potencias de 2, extendiendo un trabajo anterior en el cual se caracteriza todas las potencias de 2 que son sumas de dos números de Fibonacci.

Finalmente, estudiamos la variante de la ecuación de Brocard-Ramanujan $m!+1 = u^2$, donde u es un término de una sucesión de enteros positivos. Bajo algunas condiciones técnicas sobre la sucesión, demostramos que esta ecuación tiene a lo más un número finito de soluciones en enteros positivos m y u. Como aplicación, resolvemos la ecuación cuando u es un número Tripell. Los números Tripell, los cuales representan una generalización de los números de Pell, están definidos por la recurrencia $T_n = 2T_{n-1} + T_{n-2} + T_{n-3}$ para $n \geq 3$, con condiciones iniciales $T_0 = 0$, $T_1 = 1$ y $T_2 = 2$. Para esta última sucesión, estudiamos su valuación 2-ádica y 3-ádica con el objetivo de determinar todos los números Tripell que son factoriales. Las principales herramientas utilizadas en esta investigación son cotas inferiores para formas lineales en logaritmos de números algebraicos y una versión del método de reducción de Baker-Davenport proveniente de aproximación Diofántica. También utilizamos el método de construcción de identidades de Zhou, el cual permite construir identidades para sucesiones lineales recurrentes.

Frases y palabras clave: Problema de Pillai, número $k-$Fibonacci, número de Pell, formas lineales en logaritmos, método de reducción, valuación p-ádica, método de Zhou.

Abstract

Let us consider, for an integer $k \geq 2$, the $k-$Fibonacci sequence which starts with $0, \ldots, 0, 1$ (a total of k terms) and each term afterwards is the sum of the k preceding terms. The Fibonacci numbers are obtained for $k = 2$, while when, for example, $k = 3$, the resulting sequence is the Tribonacci sequence, and so on. The Pell sequence $\{P_n\}_{n\geq 0}$ is the second order linear recurrence defined by $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$, with initial conditions $P_0 = 0$ and $P_1 = 1$. In this thesis, we are interested in finding all integers c having at least two representations as a difference between a $k-$Fibonacci number and a Pell number. This problem, which can be seen as a variation of the well–known Pillai's problem, extends previous works concerning the cases $c = 0$ and $k = 2$. We also find all ratios of sums of two Fibonacci numbers equal to powers of 2, extending a previous work which investigated the powers of 2 which are sums of two Fibonacci numbers.

Finally, we study the variant of the Brocard-Ramanujan equation $m! + 1 = u^2$, where u is a member of a sequence of positive integers. Under some technical conditions on the sequence, we prove that this equation has at most finitely many solutions in positive integers m and u. As an application, we completely solve this equation when u is a Tripell number. The Tripell numbers are a generalization of the Pell numbers defined by the recurrence relation $T_n = 2T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 3$, with $T_0 = 0$, $T_1 = 1$ and $T_2 = 2$ as initial conditions. For this last sequence, we study its 2-adic and 3-adic valuation to determine all Tripell numbers which are factorials. The primary tools used in our investigation are lower bounds for linear forms in logarithms of algebraic numbers and a version of the Baker-Davenport reduction method from Diophantine approximation. We also make use of Zhou's method of constructing identities for linear recurrence sequences.

Keywords: Pillai's problem, $k-$Fibonacci number, Pell number, linear form in logarithms, reduction method, $p-$adic valuation, Zhou's method.

Research products

List of publications

This thesis is based on the results of the following journal papers:

1. J. J. Bravo, M. Díaz and J. L. Ramírez, *The* 2*-adic and* 3*-adic valuation of the Tripell sequence and an application*, Turkish J. Math., **44** (2020), no. 1, 131–141.
2. J. J. Bravo, M. Díaz and F. Luca, *Ratios of sums of two Fibonacci numbers equal to powers of 2*, Math. Commun., **25** (2020), no. 2, 185–199.
3. J. J. Bravo, M. Díaz and J. L. Ramírez, *On a variant of the Brocard–Ramanujan equation and an application*, Publ. Math. Debrecen, **98** (2021), no. 1, 243–253.
4. J. J. Bravo, M. Díaz and C. A. Gómez, *Pillai's problem with* $k-$*Fibonacci and Pell numbers*, J. Difference Equ. Appl., **27** (2021), no. 10, 1434-1455.

Talks

The following talks follow from the results presented in this thesis:

– *The* 2*-adic and* 3*-adic valuation of the Tripell sequence and an application*, XXII Congreso Colombiano de Matemáticas, June 10–14, 2019.

– *The* 2*-adic and* 3*-adic valuation of the Tripell sequence and an application*, Ciclo de Conferencias en Teoría de Números y Álgebra, Universidad del Valle, Cali Colombia, July 10–11, 2019.

– *Sobre una variante de la ecuación de Brocard Ramanujan*, Seminario ALTENUA Online, October 8, 2020.

- *On a variant of the Brocard-Ramanujan equation*, Sespomat-Unicauca, April 19, 2021.

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Chapter 1

Introduction

There are a lot of linear recurrence sequences which are used in number theory. For instance, the Fibonacci sequence $\{F_n\}_{n\geq 0}$ is one of the most famous and curious numerical sequences in mathematics and has been widely studied in the literature. This sequence can be generalized in several ways, some of them preserving the initial conditions and altering the recurrence relation slightly, others preserving the recurrence relation and altering the initial conditions. In this thesis, we consider, for an integer $k \geq 2$, a generalization of the Fibonacci sequence called the $k-$generalized Fibonacci sequence or, for simplicity, the $k-$*Fibonacci sequence* $F^{(k)} = \{F_n^{(k)}\}_{n\geq 2-k}$ which is defined by the recurrence relation

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)} \quad \text{for} \quad n \geq 2,$$

with initial conditions

$$F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_0^{(k)} = 0 \quad \text{and} \quad F_1^{(k)} = 1.$$

Note that this generalization is in fact a family of sequences where each new choice of k produces a distinct sequence. For example, the usual Fibonacci sequence is obtained for $k = 2$, and for $k = 3$ we have the Tribonacci sequence. The problem of determining all integer solutions to Diophantine equations with Fibonacci numbers and its generalizations has gained a considerable amount of interest among the mathematicians and there is a very broad literature on this subject. For the beauty and rich applications of these numbers and their relatives one can see Koshy's book [50].

In this thesis, we also consider the Pell sequence $P = \{P_n\}_{n\geq 0}$ defined by the recurrence

$$P_n = 2P_{n-1} + P_{n-2} \quad \text{for all} \quad n \geq 2,$$

with the initial conditions $P_0 = 0$ and $P_1 = 1$.

There are several papers in the literature dealing with Diophantine equations involving $k-$Fibonacci and Pell numbers. For example, Alekseyev [1] established that $F^{(2)} \cap P = \{0, 1, 2, 5\}$ and his result was extended to general k by Bravo, Gómez and Herrera [15], who found all generalized Fibonacci numbers which are Pell numbers. Further details about these sequences can be found, for instance, in [17, 20, 21, 50].

One of the generalizations of the Pell sequence is what we have called the *Tripell sequence* $T = \{T_n\}_{n\geq 0}$. This sequence starts with $T_0 = 0$, $T_1 = 1$, $T_2 = 2$ and each following term is given by the recurrence

$$T_n = 2T_{n-1} + T_{n-2} + T_{n-3}.$$

This thesis is divided into six chapters. This chapter is introductory, as the title suggests, and Chapter 2 gives the main tools and the preliminary results which will be used in this work. Chapter 2 consists of three independent sections. In Section 2.1, we give an overview of linear recurrence sequences and introduce the theory of constructing identities by Zhou [83] that will be used in this thesis. In Section 2.2 we study some properties of continued fractions. The aim of this section is to achieve Lemma 2.1 and Lemma 2.2. This latter lemma, which will be one of the key tools used in this thesis to reduce upper bounds, is the version given by Bravo, Gómez and Luca [17] and comes from a slight variation of a result due to Dujella and Petho from [35]. Chapter 2 concludes with Section 2.3 in which we present a brief survey of linear forms in logarithms of algebraic numbers. Specifically, we will present a result due to Matveev [59] which gives us a general lower bound for linear forms.

In Chapter 3 we apply the method of linear forms in logarithms and reduction techniques to solve a variant of the well–known Pillai's equation in the context of linear recurrence sequences. Specifically, we find all integers having at least two representations as a difference between a k-Fibonacci number and a Pell number. In Chapter 4 we also apply linear forms in logarithms to solve a Diophantine equation involving Fibonacci numbers and powers of 2.

In Chapter 5 we study the 2-adic and 3-adic valuation of the Tripell sequence and, as an application, we determine all Tripell numbers which are factorials. Finally, in Chapter 6 we study the variant of the Brocard–Ramanujan Diophantine equation $m! + 1 = u^2$, where u is a member of a sequence of positive integers.

This thesis is written in such a way that, after reading Chapter 2, the reader should be able to understand each one of the following chapters separately. In what follows, we give an overview of the contents of Chapters 3 to 6.

Let us suppose that a, b and c are fixed nonzero integers and consider the exponential Diophantine equation

$$a^x - b^y = c. \tag{1.1}$$

In 1936 and again in 1945 (see [63]), Pillai formulated his famous conjecture, which states that for any fixed integer $c \geq 1$, the Diophantine equation (1.1) has only finitely many positive integer solutions (a, b, x, y) with $x, y \geq 2$. This conjeture is still open for all $c \neq 1$. The case $c = 1$ is Catalan's conjecture which was solved by Mihăilescu [60]. For references and more on the history of this problem, we refer the reader to the survey [80].

Some recent results related to equation (1.1) have been obtained by several authors in the context of linear recurrence sequences, i.e., by replacing the powers of a and b by members of linear recurrence sequences. To fix ideas, let $\{U_n\}_{n\geq 0}$ and $\{V_m\}_{m\geq 0}$ be two linear recurrence sequences of integers and consider the Diophantine equation

$$U_n - V_m = c \tag{1.2}$$

for a fixed integer c and positive integers n and m. Chim, Pink and Ziegler [28] studied equation (1.2) and proved that under some mild restrictions, there exist only finitely many integers c such that equation (1.2) has at least two distinct solutions (n, m). Then, the problem of determining all integers c having at least two representations of the form $U_n - V_m$ can be regarded as a variant of Pillai's problem. This variant was started by Ddamulira, Luca and Rakotomalala [33] with Fibonacci numbers and powers of 2. Shortly afterwards, Bravo, Luca and Yazán [23] considered the same Diophantine equation in Tribonacci numbers instead of Fibonacci numbers. In [43], Hernández, Luca and Rivera also considered this variant with Fibonacci and Pell sequences. In fact, the work of Bravo, Gómez and Herrera [15] mentioned before can be seen as a variation of Pillai's problem by taking $c = 0$. Other cases including Tribonacci, Pell, Padovan and generalized Fibonacci numbers have been also studied (see [23, 27, 32, 39, 42]).

In [11], which is reproduced in Chapter 3, we study the particular case of equation (1.2) with k-Fibonacci and Pell numbers which continues and extends the works in [15, 43] concerning the cases $c = 0$ and $k = 2$, respectively. To be more precise, we consider the Diophantine equation

$$F_n^{(k)} - P_m = c \tag{1.3}$$

for a fixed c and positive integers n, m and k with $k \geq 2$. In particular, we are interested in finding all the integers c having at least two representations of the form $F_n^{(k)} - P_m$ for some positive integers n, m and k with $k \geq 2$. We prove the following.

Theorem (Chapter 3, Theorem 3.1)**.** *All the integers c having at least two representations of the form $F_n^{(k)} - P_m$ are*

$$c \in \{0, 1, 2, 3, -4, -5, 11, 12, -14, 19, 27, 31, 56, 79, -153, 758\}.$$

Furthermore, for each c in the above set, all its representations (k, n, m) of the form $F_n^{(k)} - P_m$ with $n \geq 2$, $m \geq 1$ and $k \geq 2$ belong to the sets:

- $\{(2,2,1),(2,3,2),(2,5,3),(4,7,5),(4,3,2),(4,2,1)\}$ *for* $c = 0$.
- $\{(2,3,1),(2,4,2),(2,7,4),(3,6,4),(3,3,1)\}$ *for* $c = 1$.
- $\{(2,4,1),(2,16,9),(3,5,3),(3,4,2),(5,7,5),(5,4,2)\}$ *for* $c = 2$.
- $\{(2,5,2),(2,6,3),(4,5,3),(4,4,1),(4,6,4),(5,5,3),(5,4,1),(6,5,3),$ $(6,4,1),(6,7,5),(7,7,5),(7,5,3),(7,4,1),(8,5,3),(8,4,1),(8,7,5)\}$ *for* $c = 3$.
- $\{(2,6,4),(2,2,3),(4,5,4),(4,2,3),(5,5,4),(5,2,3),(6,5,4),(6,2,3),$ $(7,5,4),(7,2,3),(8,5,4),(8,2,3)\}$ *for* $c = -4$.
- $\{(3,7,5),(3,5,4)\}$ *for* $c = -5$ *and* $\{(3,9,6),(3,6,2)\}$ *for* $c = 11$.
- $\{(3,7,4),(3,6,1)\}$ *for* $c = 12$ *and* $\{(4,8,6),(4,6,5)\}$ *for* $c = -14$.
- $\{(2,11,6),(2,8,2)\}$ *for* $c = 19$ *and* $\{(4,8,5),(4,7,2)\}$ *for* $c = 27$.
- $\{(8,12,9),(8,7,1)\}$ *for* $c = 31$ *and* $\{(5,11,8),(5,8,3)\}$ *for* $c = 56$.
- $\{(3,10,6),(3,9,2)\}$ *for* $c = 79$ *and* $\{(8,10,8),(8,6,7)\}$ *for* $c = -153$.
- $\{(3,15,10),(3,13,7)\}$ *for* $c = 758$.

In addition, there are five parametric families (k, n, m, n_1, m_1, c) of solutions for which $c = F_n^{(k)} - P_m = F_{n_1}^{(k)} - P_{m_1}$ with $n, n_1 \geq 2$ and $m, m_1 \geq 1$. Namely

$$\begin{aligned}
&(k,3,2,2,1,0),(k,4,3,2,2,-1), \quad \textit{for all} \quad k \geq 3;\\
&(k,5,4,2,3,-4), \quad \textit{for all} \quad k \geq 4;\\
&(k,7,5,5,3,3),(k,7,5,4,1,3), \quad \textit{for all} \quad k \geq 6.
\end{aligned}$$

There are many interesting Diophantine equations that arise when one studies arithmetic properties of Fibonacci numbers. For example, it is known that $1, 2, 8$ are the only powers of 2 that appear in the Fibonacci sequence. One proof of this fact follows from Carmichael's theorem on primitive divisors [26], which states that for n greater than 12, the nth Fibonacci number F_n has at least one prime factor that is not a factor of any previous Fibonacci number. From the above, it suffices to check the first 12 terms of the Fibonacci sequence and among these values one finds all the powers of 2. In 2016, Bravo and Luca [22] extended the previous result by finding all powers of 2 which are sums of two Fibonacci numbers, i.e., they found all the solutions of the Diophantine equation

$$F_n + F_m = 2^a$$

in positive integer variables (n, m, a). Inspired by these latest research, and in order to generalize the previous results a little bit more, we consider the Diophantine equation

$$F_n + F_m = 2^a(F_r + F_s) \tag{1.4}$$

in non negative integers n, m, a, r and s. In Chapter 4, which is based on the paper [12], we prove the following theorem.

Theorem (Chapter 4, Theorem 4.1)**.** *Equation* (1.4) *has two parametric families of non-degenerate solutions* (n, m, a, r, s) *with* $n > m \geq 0$ *and* $r > s \geq 0$*, namely*

$$\begin{array}{rcl} (n, n-3, 1, n-1, 0) & : & F_n + F_{n-3} = 2F_{n-1}; \quad n \geq 3; \\ (n, n-6, 1, n-2, n-4) & : & F_n + F_{n-6} = 2(F_{n-2} + F_{n-4}), \quad n \geq 6. \end{array}$$

When $n = 4,\ 7$*, in the first and second families, we must take* $m = 2$ *(instead of* $m = 1$*), respectively. In addition, putting* $N := F_n + F_m$*, there are exactly* 12 *values of* $N = F_n + F_m$ *yielding* 21 *more sporadic solutions namely:*

$$\begin{array}{rcl} 4 & = & F_4 + F_2 = 2^2 F_2; \\ 8 & = & F_6 = 2^2 F_3 = 2^3 F_2; \\ 16 & = & F_7 + F_4 = 2^2(F_4 + F_2) = 2^3 F_3 = 2^4 F_2; \\ 18 & = & F_7 + F_5 = 2(F_6 + F_2); \\ 24 & = & F_8 + F_4 = 2^2(F_5 + F_2) = 2^3 F_4; \\ 36 & = & F_9 + F_3 = 2^2(F_6 + F_2); \\ 56 & = & F_{10} + F_2 = 2^2(F_7 + F_2) = 2^3(F_5 + F_3); \\ 60 & = & F_{10} + F_5 = 2^2(F_7 + F_3); \\ 92 & = & F_{11} + F_4 = 2^2(F_8 + F_3); \\ 144 & = & F_{12} = 2^2(F_9 + F_3) = 2^3(F_7 + F_5) = 2^4(F_6 + F_2); \\ 288 & = & F_{13} + F_{10} = 2^3(F_9 + F_3) = 2^4(F_7 + F_5) = 2^5(F_6 + F_2); \\ 1008 & = & F_{16} + F_8 = 2^4(F_{10} + F_6). \end{array}$$

In number theory, for a given prime number p, the p-adic valuation, or p-adic order, of a non-zero integer n, denoted by $\nu_p(n)$, is the exponent of the highest power of p which divides n. The p-adic order of certain linear recurrence sequences has been studied by many authors. For example, the p-adic order of the Fibonacci numbers was completely characterized by Lengyel in [52]. In 2016, Sanna [70] gave simple formulas for the p-adic order $\nu_p(u_n)$, in terms of $\nu_p(n)$ and the rank of apparition of p in $\{u_n\}_{n\geq 0}$, where $\{u_n\}_{n\geq 0}$ is a nondegenerate Lucas sequence.

However, much less is known about the behavior of the p-adic valuation of linear recurrence sequences of higher order. A particular case of linear recurrence sequences of order 3 was studied by Marques and Lengyel in [53]. They characterized the 2-adic valuation of the Tribonacci sequence $F^{(3)}$. Results on the 2-adic valuation of Tetra- and Pentanacci numbers can be found in [54]. See also [73, 82] for the behaviour of the 2-adic valuation of generalized Fibonacci numbers and some applications to certain Diophantine equations.

In Chapter 5, based on [13], we use the theory of constructing identities given by Zhou in [83] and several congruence results to partially characterize the 2-adic valuation of the Tripell sequence and fully characterize the 3-adic valuation $\nu_3(T_n)$. Our main results are the following.

Theorem (Chapter 5, Theorem 5.3)**.** *The 2-adic valuation of the nth Tripell number is given by*

$$\nu_2(T_n) = \begin{cases} 0, & \text{if } n \equiv 1,3,4,5 \pmod 7; \\ 2, & \text{if } n \equiv 9 \pmod{14}; \\ 1, & \text{if } n \equiv 2,7 \pmod{14}; \\ \nu_2(n)+1, & \text{if } n \equiv 0 \pmod{14}; \\ \nu_2(n+1)+1, & \text{if } n \equiv 13 \pmod{14}. \end{cases}$$

If $n \equiv 6 \pmod{14}$, then $\nu_2(T_n) = \nu_2(n)+1$ except when $n \equiv 1280 \pmod{1792}$ or, equivalently, when n is of the form

$$n = 14(t2^7 + 2^6 + 2^4 + 2^3 + 2 + 1) + 6 = 1792t + 1280 \quad \text{with} \quad t \geq 0.$$

Theorem (Chapter 5, Theorem 5.4)**.** *The 3-adic valuation of the nth Tripell number is given by*

$$\nu_3(T_n) = \begin{cases} 0, & \text{if } n \equiv 1,2,3,4 \pmod 6; \\ \nu_3(n), & \text{if } n \equiv 0 \pmod 6; \\ \nu_3(n+1), & \text{if } n \equiv 5 \pmod 6. \end{cases}$$

And, as an application, we determine all Tripell numbers which are factorials.

Theorem (Chapter 5, Theorem 5.5)**.** *The only solutions of the Diophantine equation*

$$T_n = m! \tag{1.5}$$

in positive integers n, m *are* $(n, m) \in \{(1, 1), (2, 2)\}$.

Results on p-adic valuation of terms of recurrence linear sequences have been also used by several researchers in number theory to solve variants of the Brocard–Ramanujan Diophantine equation. The Brocard–Ramanujan problem [24, 25, 67, 68], a famously unsolved problem (see [40]), deals with finding the integer solutions to the equation

$$m! + 1 = n^2. \tag{1.6}$$

It is expected that the only solutions are $(m, n) = (4, 5), (5, 11), (7, 71)$. Computations by Berndt and Galway [8] showed that there are no other solution in the range $n < 10^9$. In 1993, Overholt [62] proved that the weak form of Szpiro's conjecture implies that equation (1.6) has only finitely many solutions. The weak form of Szpiro's conjecture is a special case of the ABC conjecture and asserts that there exists a constant s such that if A, B, and C are positive integers satisfying $A + B = C$ with $\gcd(A, B) = 1$, then $C \leq N(ABC)^s$, where $N(k)$ is the product of all primes dividing k taken without repetition.

Some variations of equation (1.6) have been considered by various authors and we refer the reader to [29, 30, 31, 55] and references therein for additional information and history. A generalization of the Brocard–Ramanujan problem was investigated by Berend and Osgood (see [7]), who showed that if $P \in \mathbb{Z}[x]$ is of degree at least 2, then the density of the set of positive integers m such that

$$P(x) = m!$$

has a solution x, is zero.

Variants of (1.6) involving linear recurrences have also been studied. For example, Marques [57] investigated the Fibonacci version of the Brocard-Ramanujan Diophantine equation, namely the equation

$$F_m F_{m+1} \cdots F_{m+k-1} + 1 = F_n^2.$$

Szalay [75] and Pongsriiam [66] worked on another version of the Brocard–Ramanujan problem with Fibonacci, Lucas and balancing numbers, extending the result of Marques

[57]. Taşci and Sevgi [76] studied Pell and Pell-Lucas numbers associated with the Brocard–Ramanujan equation, while Pink and Szikszai [65] investigated the Brocard–Ramanujan problem with Lucas and associated Lucas sequences.

In this research, we study the variant of the Brocard–Ramanujan Diophantine equation

$$m! + 1 = u_n^2, \tag{1.7}$$

where $\{u_n\}_{n\geq 0}$ is a sequence of positive integers. Under some technical conditions on the sequence, we prove that equation (1.7) has at most finitely many solutions in positive integers m and n. In [14], which is reproduced in Chapter 6, we prove the following result.

Theorem (Chapter 6, Theorem 6.1)**.** *Let $\{u_n\}_{n\geq 0}$ be a sequence of positive integers such that*

$$n = O(\log u_n).$$

Let p be a prime and assume that

$$\nu_p(u_n + 1) = O(n^{C_1}) \quad \text{and} \quad \nu_p(u_n - 1) = O(n^{C_2})$$

for some constants C_1 and C_2 with $\max\{C_1, C_2\} < 1$. Then, the Diophantine equation (1.7) has only a finite number of solutions in non-negative integers m and n.

Our method to prove the above theorem shows how to extract an upper bound for the variables of the Diophantine equation. We conclude Chapter 6 by studying equation (1.7) when u_n is a Tripell number. We prove the following result.

Theorem (Chapter 6, Theorem 6.2)**.** *The only solution of the Diophantine equation*

$$m! + 1 = T_n^2, \tag{1.8}$$

in non-negative integers m and n, is $(m, n) = (4, 3)$.

This completes the sketch of my thesis.

Chapter 2

Preliminaries

In this chapter, we give some preliminaries that will be useful for this research. The first section is devoted to the main properties of linear recurrence sequences used in this thesis. In the second part we recall some theory of Diophantine approximation and continued fractions which are required in this investigation. We conclude this chapter with results about lower bounds for linear forms in logarithms of algebraic numbers.

2.1 Linear recurrence sequences

Linear recurrence sequences have interesting properties and have been a central part of number theory for many years. Their study is plainly of intrinsic interest. We will refer to some theorems of the multitude of fundamental results that have been proved in recent years.

Definition 2.1. *A linear recurrence sequence of order $k \geq 1$ is a sequence $\{w_n\}_{n\geq 0}$ which satisfies a relation of the form*

$$w_n = a_1 w_{n-1} + a_2 w_{n-2} + \cdots + a_k w_{n-k} \tag{2.1}$$

for all $n \geq k$, where $a_1, \ldots, a_k$ are constants with $a_k \neq 0$. The values $w_0, w_1, \ldots, w_{k-1}$ are not all zero and are called the initial conditions of the sequence. We say that $\{w_n\}_{n\geq 0}$ is simple if its characteristic polynomial

$$\begin{aligned} f(x) &= x^k - a_1 x^{k-1} - a_2 x^{k-2} - \cdots - a_{k-1} x - a_k \\ &= (x - \alpha_1)^{r_1} (x - \alpha_2)^{r_2} \cdots (x - \alpha_h)^{r_h}, \end{aligned} \tag{2.2}$$

where $r_1 + r_2 + \cdots + r_h = k$, has only simple roots in a splitting field over the base field. We call $\alpha_1, \alpha_2, \ldots, \alpha_h$ the roots of the sequence $\{w_n\}_{n\geq 0}$. A recurrence sequence of order 2 is called binary; one of order 3 ternary.

Definition 2.2. *Define $\Omega(f(x))$ to be the set of all sequences $\{w_n\}_{n\geq 0}$ satisfying the recurrence (2.1).*

One may ask if there exists a closed formula that produces the numbers w_n defined by (2.1). Regarding this, it is well–known that for all n,

$$w_n = \sum_{m=1}^{h} N(n,m)\alpha_m^n, \tag{2.3}$$

where

$$N(n,m) = A_1^{(m)} + A_2^{(m)} n + \cdots + A_{r_m}^{(m)} n^{r_m - 1} = \sum_{i=0}^{r_m - 1} A_{i+1}^{(m)} n^i, \tag{2.4}$$

and each $A_i^{(m)}$ is a constant determined by the initial conditions of $\{w_n\}_{n\geq 0}$. Expression (2.3) is sometimes called the Binet type formula for the sequence $\{w_n\}_{n\geq 0}$.

If $\{w_n\}_{n\geq 0}$ is simple, then

$$w_n = A_1\alpha_1^n + A_2\alpha_2^n + \cdots + A_k\alpha_k^n \quad \text{for all} \quad n \geq 0.$$

In particular, for a binary simple sequence with distinct roots α and β, the Binet formula is given by

$$w_n = C\alpha^n + D\beta^n \quad \text{for all} \quad n \geq 2,$$

where $w_0 = C + D$ and $w_1 = C\alpha + D\beta$.

In recent years, many researchers in number theory and combinatorics have focused their attention on finding identities involving terms of certain linear recurrence sequences. In [83], Zhou introduces his *Theory of Constructing Identities*, or TCI, which writers have found to be remarkably useful to subtract a considerable amount of identities. Basically, it shows how to use certain kinds of polynomial congruences to prove identities for linear recurrence sequences. Howard and Saidak in [46] gave a proof of TCI involving the Binet formula of the involved numbers. The result is as follows.

Theorem 2.1 (TCI)**.** *For all $i = 0, \ldots,$ and $j = 0, \ldots,$ let n_i and p_j be arbitrary integers and let d_i and e_j be arbitrary complex numbers. Let $f(x)$ be given by (2.2) and suppose*

$$\sum_{i=0}^{s} d_i x^{n_i} \equiv \sum_{j=0}^{t} e_j x^{p_j} \pmod{f(x)}. \tag{2.5}$$

Then, for all $\{w_n\}_n$ *in* $\Omega(f(x))$ *we have*

$$\sum_{i=0}^{s} d_i w_{n_i} = \sum_{j=0}^{t} e_j w_{p_j}. \tag{2.6}$$

Conversely, if (2.6) *holds for all* $\{w_n\}_{n\geq 0}$ *in* $\Omega(f(x))$*, then* (2.5) *holds.*

We next present the proof of Theorem 2.1 given by Howard and Saidak in [46].

Proof. First, assume that (2.5) holds, and let

$$F(x) = \sum_{i=0}^{s} d_i x^{n_i} - \sum_{j=0}^{t} e_j x^{p_j}. \tag{2.7}$$

Then, $F(x) \equiv 0 \pmod{f(x)}$ and α_m is a zero of $F(x)$ of multiplicity at least r_m . Thus,

$$F(\alpha_m) = F'(\alpha_m) = \cdots = F^{(r_m - 1)}(\alpha_m) = 0.$$

Now define $F_u(x)$ inductively by $F_0(x) = F(x)$, and for $u \geq 1$

$$F_u(x) = xF'_{u-1}(x) = \sum_{i=0}^{s} d_i (n_i)^u x^{n_i} - \sum_{j=0}^{t} e_j (p_j)^u x^{p_j}. \tag{2.8}$$

Note that $F^{(u)}(\alpha_m) = 0$ for $u = 0, 1, \ldots, r_m - 1$, if and only if, $F_u(\alpha_m) = 0$ for $u = 0, 1, \ldots, r_m - 1$. Hence, for $m = 1, 2, \ldots, h$

$$A_1^{(m)} F_0(\alpha_m) = A_2^{(m)} F_1(\alpha_m) = \cdots = A_{r_m}^{(m)} F_{r_m - 1}(\alpha_m) = 0, \tag{2.9}$$

for some constants $A_i^{(m)}$ for $i = 1, \ldots, r_m$. That is, for $u = 0, 1, \ldots, r_m - 1$

$$A_{u+1}^{(m)} F_u(\alpha_m) = \sum_{i=0}^{s} d_i A_{u+1}^{(m)} (n_i)^u (\alpha_m)^{n_i} - \sum_{j=0}^{t} e_j A_{u+1}^{(m)} (p_j)^u (\alpha_m)^{p_j} = 0,$$

so

$$\sum_{i=0}^{s} d_i A_{u+1}^{(m)} (n_i)^u (\alpha_m)^{n_i} = \sum_{j=0}^{t} e_j A_{u+1}^{(m)} (p_j)^u (\alpha_m)^{p_j}. \tag{2.10}$$

Now let $\{w_n\}_{n\geq 0}$ be any sequence in $\Omega(f(x))$ with Binet formula (2.3).

In (2.10) we sum from $u = 0$ to $u = rm - 1$ to obtain

$$\sum_{u=0}^{r_m-1}\sum_{i=0}^{s} d_i A_{u+1}^{(m)}(n_i)^u(\alpha_m)^{n_i} = \sum_{u=0}^{r_m-1}\sum_{j=0}^{t} e_j A_{u+1}^{(m)}(p_j)^u(\alpha_m)^{p_j},$$

giving

$$\sum_{i=0}^{s} d_i N(n_i, m)(\alpha_m)^{n_i} = \sum_{j=0}^{t} e_j N(p_j, m)(\alpha_m)^{p_j}, \tag{2.11}$$

where $N(n, m)$ is defined by (2.4). Then

$$\sum_{m=1}^{h}\sum_{i=0}^{s} d_i N(n_i, m)(\alpha_m)^{n_i} = \sum_{m=1}^{h}\sum_{j=0}^{t} e_j N(p_j, m)(\alpha_m)^{p_j},$$

and so

$$\sum_{i=0}^{s} d_i w_{n_i} = \sum_{j=0}^{t} e_j w_{p_j}.$$

Conversely, assume that (2.6) holds for all $\{w_n\}_{n\geq 0}$ in $\Omega(f(x))$. For $m = 1, 2, \ldots, h$, we know, by (2.2), that $((\alpha_m)^n), (n(\alpha_m)^n), \ldots, (n^{r_m-1}(\alpha_m)^n)$ are all in $\Omega(f(x))$. Thus, for $u = 0, 1, \ldots, r_m - 1$,

$$F_u(\alpha_m) = \sum_{i=0}^{s} d_i(n_i)^u(\alpha_m)^{n_i} - \sum_{j=0}^{t} e_j(p_j)^u(\alpha_m)^{p_j} = 0, \qquad (m = 1, 2, \ldots, h).$$

Hence, by (2.7),

$$F(\alpha_m) = F'(\alpha_m) = \ldots = F^{(r_m-1)}(\alpha_m) = 0.$$

Consequently, $(x - \alpha_m)^{r_m}$ is a factor of $F(x)$ for $m = 1, 2, \ldots, h$, which implies that $F(x) \equiv 0 \pmod{f(x)}$. Thus

$$\sum_{i=0}^{s} d_i x^{n_i} \equiv \sum_{j=0}^{t} e_j x^{p_j} \pmod{f(x)}.$$

□

2.2 Diophantine approximation and continued fractions

Diophantine approximation is the study of how closely an algebraic number $\alpha \in \mathbb{R}$ can be approximated by a rational number p/q. The obvious measure of the accuracy of a Diophantine approximation of a real number α by a rational number p/q is $|\alpha - p/q|$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, this is always possible to an arbitrary accuracy. For such a comparison, one may want upper bounds or lower bounds of the approximation. Liouville in 1853 showed that algebraic numbers cannot be too well approximated by rationals (see [61, Theorem 3.2.1]).

Theorem 2.2 (Liouville). *Let $\alpha \in \mathbb{R}$ an algebraic number of degree $d \neq 1$. Then, there is a constant $C := C(\alpha)$ such that for all rational numbers p/q, $\gcd(p, q) = 1$, the inequality*

$$\left|\alpha - \frac{p}{q}\right| > \frac{C}{q^d}$$

holds.

By using the above result, Louville was the first to explicitly construct transcendental numbers, as observed in the following corollary.

Corollary 2.1 (Liouville). *Let $\beta \in \mathbb{R}$ be an irrational number and suppose that for all $C > 0$ and all integer $d \geq 1$, there exists $p/q \in \mathbb{Q}$ such that*

$$\left|\beta - \frac{p}{q}\right| \leq \frac{C}{q^d}.$$

Then, β is transcendental, i.e., β is not algebraic over $\mathbb{Q}$.

For long, there have been many authors who improved Liouville's result. For example Thue, Siegel, Dyson and Roth (see [36, 69, 71, 72, 77, 78]) had successively improved Liouville's original exponent d which led finally to the Thue-Siege-Roth's theorem.

Theorem 2.3. *Let $\alpha \in \mathbb{R}$ be an irrational algebraic number. For every $\epsilon > 0$, there is a constant $C := C(\epsilon, \alpha)$ such that*

$$\left|\alpha - \frac{p}{q}\right| > \frac{C}{q^{2+\epsilon}},$$

for all irreducible rationals p/q with $q > 0$.

It follows that every irrational algebraic number α has approximation exponent equal to 2. This means that, for every $\epsilon > 0$, the inequality

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q^{2+\epsilon}},$$

can have only finitely many solutions in coprime integers p and q.

Continued fractions also play an important role nowadays in number theory. They constitute one of the best tools for new discoveries in the theory of numbers and in the field of Diophantine approximations. For the purposes of this section, we will focus only on simple continued fractions. They have the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}} \qquad \text{or} \qquad a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}},$$

where a_0 is an integer and $a_1, a_2, \ldots$ are positive integers. In the first case we have a finite simple continued fraction and in the second case we have an infinite simple continued fraction. The above expressions are usually denoted by $[a_0, a_1, \ldots, a_n]$ or by $[a_0, a_1, a_2, \ldots]$ for a finite or infinite simple continued fraction, respectively.

One can easily see that every finite simple continued fraction is a rational number. Conversely, using the Euclidean algorithm, every rational number can be represented as a finite simple continued fraction.

The number $C_n := [a_0, a_1, \ldots, a_n]$ is called the nth convergent of the continue fraction $[a_0, a_1, a_2, \ldots]$. Clearly, it is a rational number. It is important to develop a systematic way of computing these convergents. To do this we write:

$$\begin{aligned}
C_0 &= [a_0] = \frac{a_0}{1} = \frac{p_0}{q_0}, \\
C_1 &= [a_0, a_1] = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1} = \frac{p_1}{q_1}, \\
&\vdots \\
C_n &= [a_0, a_1, \ldots, a_n] = a_0 + \frac{1}{[a_1, a_2, \ldots, a_n]} = [a_0, [a_1, a_2, \ldots, a_n]] = \frac{p_n}{q_n}.
\end{aligned}$$

It is well–know that the sequences of numerators $\{p_n\}_{n\geq 0}$ and denominators $\{q_n\}_{n\geq 0}$ satisfy the following recursive formulas:

$$\begin{cases} p_0 = 0, \\ p_1 = a_0 a_1 + 1, \\ \quad \vdots \\ p_n = a_n p_{n-1} + p_{n-2} \qquad \text{for all} \quad n \geq 2, \end{cases}$$

and

$$\begin{cases} q_0 = 1, \\ q_1 = a_1, \\ \quad \vdots \\ q_n = a_n q_{n-1} + q_{n-2} \qquad \text{for all} \quad n \geq 2. \end{cases}$$

It follows that the nth convergent of $[a_0, a_1, a_2, \ldots]$ is given by

$$C_n = \frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}}.$$

Proposition 2.1. *The following properties hold.*

(a) $(p_k, q_k) = 1$;

(b) $p_k q_{k-1} - p_{k-1} q_k = (-1)^{k-1}$ *for* $k \geq 1$;

(c) $C_k - C_{k-1} = \frac{(-1)^{k-1}}{q_k q_{k-1}}$ *for* $k \geq 1$;

(d) $C_k - C_{k-2} = \frac{(-1)^k a_k}{q_k q_{k-2}}$ *for* $k \geq 2$;

(e) *For any positive real number* x*, we have*

$$[a_0, a_1, \ldots, a_{k-1}, x] = \frac{x p_{k-1} + p_{k-2}}{x q_{k-1} + q_{k-2}}.$$

It follows from the above proposition that the sequence $C_1, C_3, C_5, \ldots$ it is a decreasing monotonic sequence, while $C_0, C_2, C_4, \ldots$ is an increasing monotone sequence. However, the even and odd convergents are still arbitrarily close. In addition to this, we have that $C_0 < C_2 < \cdots < C_3 < C_1$ and so both sequences $\{C_{2n}\}_{n \geq 0}$ and $\{C_{2n+1}\}_{n \geq 0}$ are convergent and converge to the same value. So that

$$\lim_{n \to \infty} C_n$$

exists. This means that every infinite simple continued fraction converges, so we can define:

Definition 2.3. *Let $a_0, a_1, a_2, \ldots$ be a sequence of integers with $a_i > 0$ for $i > 0$. Then*

$$[a_0, a_1, a_2, \ldots] := \lim_{n\to\infty} C_n = \lim_{n\to\infty} \frac{p_n}{q_n} = \lim_{n\to\infty} [a_0, a_1, \ldots, a_n].$$

With the definitions we made above one can prove the following result.

Theorem 2.4. *If $\alpha = [a_0, a_1, \ldots]$ is an infinite simple continued fraction, then α is an irrational number.*

From the above, we have that every infinite simple continued fraction is an irrational number. Conversely, every irrational number can be represented as an infinite simple continued fraction, as we see below.

Theorem 2.5. *Let $\alpha = \alpha_0$ be a positive irrational number and define the sequence $\{a_i\}_{i\geq 0}$ recursively as follows:*

$$a_i := \lfloor \alpha_i \rfloor \quad \textit{and} \quad \alpha_{i+1} := \frac{1}{\alpha_i - a_i} \quad \textit{for} \quad i = 0, 1, \ldots$$

Then, $\alpha = [a_0, a_1, \ldots]$ is a representation of α as a simple continued fraction.

One can also prove that the convergents of the continued fraction expansion of an irrational number provide the *best* rational approximations, in the following sense.

Theorem 2.6. *Let x be in irrational number and let $C_n = p_n/q_n$ be the nth convergent of an infinite simple continued fraction. If $a, b \in \mathbb{Z}$ and $1 \leq b \leq q_n$, then*

$$|x - C_n| \leq \left| x - \frac{a}{b} \right|.$$

This means that, among all rational numbers with denominator no larger than q_n, C_n is the closest number to x.

The following lemma (see Theorem 8.2.4 and top of page 287 in [61]) is also an important result when studying continued fractions because it tells us that the best approximations of irrational numbers by rational numbers are given by their convergents. Let us see.

Lemma 2.1 (Legendre)**.** *Let $\tau = [a_0, a_1, a_2, \ldots]$ be the continued fraction expansion of a real number τ, and let x, y be integers such that*

$$\left| \tau - \frac{x}{y} \right| < \frac{1}{2y^2}. \tag{2.12}$$

Then, $x/y = p_k/q_k$ is a convergent of τ. Furthermore,

$$\left|\tau - \frac{x}{y}\right| \geq \frac{1}{(a_{k+1}+2)y^2}. \tag{2.13}$$

In the course of future calculations, we will obtain some upper bounds for the variables involved in certain Diophantine equations. However, these upper bounds are very large that we need to reduce them to size that can be more easily handled. For this purpose we can use either the previous result or the next lemma which is a slight variation of a result due to Dujella and Pethő (see [35, Lemma 5a]). In this thesis we shall use the version given by Bravo, Gómez and Luca (see [17, Lemma 1]). First we need to introduce some notation.

For a real number X, we write $||X|| := \min\{|X-n| : n \in \mathbb{Z}\}$ for the distance from X to the nearest integer.

Lemma 2.2. *Let M be a positive integer, p/q be a convergent of the continued fraction of the irrational number τ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. If $\varepsilon := ||\mu q|| - M||\tau q|| > 0$, then there is no solution to the inequality*

$$0 < |u\tau - v + \mu| < AB^{-w},$$

in positive integers u, v and w with

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

We finally note that the above lemma cannot be applied when μ is a linear combination of 1 and τ, since then $\varepsilon < 0$. In this case, we use the criterion of Legendre given by Lemma 2.1.

2.3 Linear forms in logarithms

In this short section, we briefly describe what nowadays is called the theory of linear forms in logarithms of algebraic numbers. It will be sufficient for us to give at the end the general lower bound for linear forms in logarithms due to Matveev [59].

Let $\alpha_1, \ldots, \alpha_n$ be complex numbers. We say that $\alpha_1, \ldots, \alpha_n$ are linearly dependent over $\mathbb{Q}$ if there exist $A_1, \ldots, A_n \in \mathbb{Z}$ not all zero such that

$$A_1\alpha_1 + \cdots + A_n\alpha_n = 0.$$

We say that $\alpha_1, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}$ if they are not linearly dependent over $\mathbb{Q}$.

At a conference in Paris in 1900, the German mathematician David Hilbert presented his famous list of unsolved problems in mathematics. Hilbert's seventh problem concerns powers of algebraic numbers and in particular with the following question: *If $\alpha \neq 0, 1$ is algebraic and β is algebraic but no rational, is α^β transcendental or at least no rational?* As specific examples he mentioned $2^{\sqrt{2}}$ and $e^\pi = i^{-2i}$.

The problem was resolved independently by Gelfond and Schneider in 1934 showing that the answer is affirmative. Their result is the following

Theorem 2.1. *If α and β are non-zero algebraic numbers with $\log \alpha$ and $\log \beta$ linearly independent over $\mathbb{Q}$, then $\log \alpha$ and $\log \beta$ are linearly independent over the algebraic numbers.*

We remark that the previous theorem implies the following result which establishes the transcendence of a large class of numbers.

Theorem 2.2. *If α and β are algebraic numbers with $\alpha \neq 0, 1$, and $\beta \notin \mathbb{Q}$, then α^β is transcendental.*

To see why Theorem 2.1 implies Theorem 2.2 we put $\beta' = \alpha^\beta = e^{\beta \log \alpha}$ and suppose that β' is an algebraic number. Then, $\log \alpha$ and $\log \beta'$ are linearly dependent over the algebraic numbers. By Theorem 2.1, $\log \alpha$ and $\log \beta'$ are also linearly dependent over $\mathbb{Q}$. Thus, there exist $A_1, A_2 \in \mathbb{Z}$ not both zero such that $A_1 \log \alpha + A_2 \log \beta' = 0$, so that $A_1 + \beta A_2 = 0$. But this is not possible since $\beta \notin \mathbb{Q}$.

Gelfond emphasized the importance of getting a generalization of this statement to more than two logarithms. This problem was solved in 1966 by Baker [2, 3] who established the following generalization.

Theorem 2.3. *If $\alpha_1, \ldots, \alpha_n$ are non-zero algebraic numbers with $\log \alpha_1, \ldots, \log \alpha_n$ linearly independent over $\mathbb{Q}$, then $1, \log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over the algebraic numbers.*

Remark 2.1. *Accordingly to Theorem 2.3, any expression of the form*

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n,$$

where $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$, are non-zero algebraic numbers and β_0 is algebraic, vanishes only in trivial cases.

Definition 2.4. *A linear form in logarithms of algebraic numbers is an expression of the form*

$$\Lambda = \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n,$$

where the α's and the β's denote complex algebraic numbers.

From the discussion above, we can ask about lower bounds for nonzero linear forms in logarithms. In 1935, Gelfond was the first to publish a lower bound for a linear form in two logarithms. He proved that if $n = 2$ and β_1 and β_2 are rational integers, say $\beta_1 = p_1/q_1$ and $\beta_2 = p_2/q_2$, then for every $\epsilon > 0$,

$$|\Lambda| > C(\alpha_1, \alpha_2, \epsilon) \exp(-\log B)^{3+\epsilon},$$

where $B = \max\{p_1, p_2, q_1, q_2\}$. Almost 30 years later, Baker wrote a sequence of 4 papers from 1966 to 1968 [2, 3, 4, 5] dealing with any number of logarithms. In particular, Baker established, among many other things, the following lower bound.

As usual, define the height of an algebraic number as the maximum of the absolute values of the relatively prime integer coefficients of its minimal defining polynomial.

Theorem 2.4. *Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers from $\mathbb{C}$ different from $0, 1$ such that $\log \alpha_1, \ldots, \log \alpha_n$ and $2\pi i$ are linearly independent over $\mathbb{Q}$. Suppose that $K > n + 1$ and let d be any positive integer. Then*

$$|\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n| > Ce^{-(\log H)^K} \tag{2.14}$$

for all algebraic numbers $\beta_1, \ldots, \beta_n$, not all zero, with degrees at most d, where H denotes the maximum of the heights of $\beta_1, \ldots, \beta_n$, and $C > 0$ is an effectively computable constant depending on $n, d, K, \alpha_1, \ldots, \alpha_n$.

In 1967 Baker obtained (2.14) for any $K > 2n + 1$ when only $\log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over $\mathbb{Q}$. Baker also raised a result for lower limits of inhomogeneous forms

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n,$$

and obtained (2.14) for any $K > n$, assuming that $\log \alpha_1, \ldots, \log \alpha_n$ or $\beta_1, \ldots, \beta_n$ are linearly independent over $\mathbb{Q}$. The special case when $\beta_1, \ldots, \beta_n$ are rational integers was presented by Baker in the following form.

Theorem 2.5 (Baker, 1975)**.** *Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers from $\mathbb{C}$ different from $0, 1$, and let $\beta_1, \ldots, \beta_n$ be rational integers such that*

$$\Lambda = \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n \neq 0.$$

Then,

$$|\Lambda| > (eB)^{-C},$$

where B denotes the maximum of the heights of $\beta_1, \ldots, \beta_n$ and C is an effectively computable constant depending only on n and on $\alpha_1, \ldots, \alpha_n$.

The previous theorem leads to the following result whose proof can be found in [37].

Corollary 2.2. *Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers from $\mathbb{C}$ different from $0, 1$, and let $\beta_1, \ldots, \beta_n$ be rational integers such that $\alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n} \neq 1$. Then*

$$|\alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n} - 1| > (eB)^{-C'},$$

where B denotes the maximum of the heights of $\beta_1, \ldots, \beta_n$ and C' is an effectively computable constant depending only on n and on $\alpha_1, \ldots, \alpha_n$.

After Baker introduced his results many mathematicians worked to refine them, but it was not until 2000 that Matveev in [59] found a better lower bound of the Baker–type for a nonzero linear form in logarithms of algebraic numbers, which we present below.

We start by recalling a definition. Let η be an algebraic number of degree d with minimal primitive polynomial over the integers

$$a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} (x - \eta^{(i)}),$$

where the leading coefficient a_0 is positive and the $\eta^{(i)}$'s are the conjugates of η. Then the *logarithmic height* of η is given by

$$h(\eta) := \frac{1}{d} \left(\log a_0 + \sum_{i=1}^{d} \log \left(\max\{|\eta^{(i)}|, 1\} \right) \right).$$

In particular, if $\eta = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then $h(\eta) = \log \max\{|p|, q\}$.

The following are some of the properties [79, Property 3.3] of the logarithmic height function $h(\cdot)$, which will be used in the remaining of this document without reference. For η, γ algebraic numbers and $s \in \mathbb{Z}$, we have

$$\begin{aligned} h(\eta \pm \gamma) &\leq h(\eta) + h(\gamma) + \log 2, \\ h(\eta \gamma^{\pm 1}) &\leq h(\eta) + h(\gamma), \\ h(\eta^s) &= |s| h(\eta). \end{aligned} \tag{2.15}$$

With this notation, Matveev (see [59]) proved the following deep theorem which will be used throughout this thesis.

Theorem 2.6 (Matveev's theorem)**.** *Let $\gamma_1, \ldots, \gamma_t$ be positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D, and let $b_1, \ldots, b_t$ be nonzero integers, and assume that*

$$\Lambda := \gamma_1^{b_1} \ldots \gamma_t^{b_t} - 1 \neq 0.$$

Then,

$$\log|\Lambda| > -1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1 + \log D)(1 + \log B)A_1 \ldots A_t,$$

where

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and

$$A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\} \quad \textit{for all} \quad i = 1, \ldots, t.$$

Chapter 3

Pillai's problem with k-Fibonacci and Pell numbers

In this chapter, we find all integers c having at least two representations as a difference between a k-Fibonacci number and a Pell number. This research continues and extends the previous work of [15, 42, 43].

3.1 Introduction

In 1936 and again in 1945 (see [63]), Pillai formulated his famous conjecture, which states that for any fixed integer $c \geq 1$, the Diophantine equation

$$a^x - b^y = c, \tag{3.1}$$

where a, b are fixed nonzero integers, has only finitely many positive integer solutions (a, b, x, y) with $x, y \geq 2$. This conjeture is still open for all $c \neq 1$. The case $c = 1$ is Catalan's conjecture which was solved by Mihăilescu [60].

The work started by Pillai was pursued in 1936 by Herschfeld [44, 45] who proved that equation (3.1) has finitely many solutions in the particular case $(a, b) = (2, 3)$. Pillai [63, 64] extended Herschfeld's result to general (a, b) with $\gcd(a, b) = 1$ and $a > b \geq 2$. Specifically, Pillai showed that there exists a positive integer $c_0(a, b)$ such that, for $|c| > c_0(a, b)$, equation (3.1) has at most one positive integer solution (x, y). In particular, he conjectured that if $(a, b) = (2, 3)$ and $|c| > 13$, then equation (3.1) has at most one solution. This conjecture was confirmed by Stroeker and Tijdeman [74] and their result was further improved by Bennett [6], who showed that equation (3.1) has at most two solutions for fixed a, b and c with $a, b > 2$.

Some recent results related to equation (3.1) have been obtained by several authors in the context of linear recurrence sequences, i.e., by replacing the powers of a and b by members of linear recurrence sequences. To make things clear, let $\{U_n\}_{n\geq 0}$ and $\{V_m\}_{m\geq 0}$ be two linear recurrence sequences of integers and consider the Diophantine equation

$$U_n - V_m = c \tag{3.2}$$

for a fixed integer c and positive integers n and m. Chim, Pink and Ziegler [28] studied equation (3.2) and proved that under some mild restrictions, there exist only finitely many integers c such that equation (3.2) has at least two distinct solutions (n, m). Then, the problem of determining all integers c having at least two representations of the form $U_n - V_m$ can be regarded as a variant of Pillai's problem. This variant was started by Ddamulira, Luca and Rakotomalala [33] with Fibonacci numbers and powers of 2. In [43], Hernández, Luca and Rivera also considered this variant with Fibonacci and Pell sequences. Other cases including Tribonacci, Pell, Padovan and generalized Fibonacci numbers have been also studied (see [23, 27, 32, 39, 42]).

We study the particular case of equation (3.2) with k-Fibonacci and Pell numbers. To be more precise, we consider the Diophantine equation

$$F_n^{(k)} - P_m = c \tag{3.3}$$

for a fixed c and positive integers n, m and k with $k \geq 2$. In particular, we are interested in finding all the integers c having at least two representations of the form $F_n^{(k)} - P_m$ for some positive integers n, m and k with $k \geq 2$. It should be noted that our investigation extends the previous works in [15, 43] concerning the cases $c = 0$ and $k = 2$, respectively. In addition to this, since the first $k + 1$ nonzero terms in $F^{(k)}$ are powers of 2, indeed we have that $F_n^{(k)} = 2^{\max\{0, n-2\}}$ for $1 \leq n \leq k + 1$, our work can be also regarded as an extension of the work in [42] that searched for all integers c having at least two representations of the form $P_n - 2^m$.

If the k-Fibonacci number involved in (3.3) equals 1, we then assume that its index is 2 in order to avoid trivial parametric families such as $F_1^{(k)} - P_m = F_2^{(k)} - P_m$. Thus, we assume that $n \geq 2$.

Our main result is the following.

Theorem 3.1. *All the integers c having at least two representations of the form $F_n^{(k)} - P_m$ are*

$$c \in \{0, 1, 2, 3, -4, -5, 11, 12, -14, 19, 27, 31, 56, 79, -153, 758\}.$$

Furthermore, for each c in the above set, all its representations (k, n, m) of the form $F_n^{(k)} - P_m$ with $n \geq 2$, $m \geq 1$ and $k \geq 2$ belong to the sets:

- $\{(2,2,1), (2,3,2), (2,5,3), (4,7,5), (4,3,2), (4,2,1)\}$ *for* $c = 0$.
- $\{(2,3,1), (2,4,2), (2,7,4), (3,6,4), (3,3,1)\}$ *for* $c = 1$.
- $\{(2,4,1), (2,16,9), (3,5,3), (3,4,2), (5,7,5), (5,4,2)\}$ *for* $c = 2$.
- $\{(2,5,2), (2,6,3), (4,5,3), (4,4,1), (4,6,4), (5,5,3), (5,4,1), (6,5,3),$ $(6,4,1), (6,7,5), (7,7,5), (7,5,3), (7,4,1), (8,5,3), (8,4,1), (8,7,5)\}$ *for* $c = 3$.
- $\{(2,6,4), (2,2,3), (4,5,4), (4,2,3), (5,5,4), (5,2,3), (6,5,4), (6,2,3),$ $(7,5,4), (7,2,3), (8,5,4), (8,2,3)\}$ *for* $c = -4$.
- $\{(3,7,5), (3,5,4)\}$ *for* $c = -5$ *and* $\{(3,9,6), (3,6,2)\}$ *for* $c = 11$.
- $\{(3,7,4), (3,6,1)\}$ *for* $c = 12$ *and* $\{(4,8,6), (4,6,5)\}$ *for* $c = -14$.
- $\{(2,11,6), (2,8,2)\}$ *for* $c = 19$ *and* $\{(4,8,5), (4,7,2)\}$ *for* $c = 27$.
- $\{(8,12,9), (8,7,1)\}$ *for* $c = 31$ *and* $\{(5,11,8), (5,8,3)\}$ *for* $c = 56$.
- $\{(3,10,6), (3,9,2)\}$ *for* $c = 79$ *and* $\{(8,10,8), (8,6,7)\}$ *for* $c = -153$.
- $\{(3,15,10), (3,13,7)\}$ *for* $c = 758$.

In addition, there are five parametric families (k, n, m, n_1, m_1, c) of solutions for which $c = F_n^{(k)} - P_m = F_{n_1}^{(k)} - P_{m_1}$ with $n, n_1 \geq 2$ and $m, m_1 \geq 1$. Namely

$$\begin{array}{l} (k,3,2,2,1,0), (k,4,3,2,2,-1), \quad \textit{for all} \quad k \geq 3; \\ (k,5,4,2,3,-4), \quad \textit{for all} \quad k \geq 4; \\ (k,7,5,5,3,3), (k,7,5,4,1,3), \quad \textit{for all} \quad k \geq 6. \end{array}$$

3.2 Preliminary inequalities

The following lemma, which was given by Guzmán and Luca in [41], is useful in the proof of the main theorem.

Lemma 3.1. *Let T be a real number and let $m \geq 1$ be and integer. If $T > (4m^2)^m$ and $T > x/(\log x)^m$, then*

$$x < 2^m T(\log T)^m.$$

3.2.1 The $k-$Fibonacci sequence

It is known (see for example, [81]) that the characteristic polynomial of the k-Fibonacci sequence $F^{(k)}$, namely

$$\Psi_k(x) = x^k - x^{k-1} - \cdots - x - 1 = (x - \alpha_1) \cdots (x - \alpha_k),$$

is irreducible over $\mathbb{Q}$ and has just one real root outside the unit circle $\alpha = \alpha_1$; the other roots are strictly inside the unit circle. It turns out that the root α, satisfies (see [81])

$$2(1 - 2^{-k}) < \alpha < 2.$$

We now consider, for an integer $k \geq 2$, the function

$$f_k(z) = \frac{z - 1}{2 + (k+1)(z-2)} \quad \text{for} \quad z \in \mathbb{C}.$$

With this notation, one can easily prove (see [17]) that the following inequalities hold

$$1/2 < f_k(\alpha) < 3/4 \quad \text{and} \quad |f_k(\alpha_i)| < 1 \quad \text{for} \quad 2 \leq i \leq k.$$

Dresden and Du proved in [34] that

$$F_n^{(k)} = \sum_{i=1}^{k} f_k(\alpha_i)\alpha_i^{n-1} \quad \text{holds for all} \quad n \geq 1 \quad \text{and} \quad k \geq 2. \tag{3.4}$$

The above expression is usually known as the "Binet–like" formula for $F^{(k)}$. It was also proved in [34] that the inequality

$$|F_n^{(k)} - f_k(\alpha)\alpha^{n-1}| < 1/2 \tag{3.5}$$

holds for all $n \geq 2-k$, which shows that the contribution of the roots which are inside the unit circle to the formula (3.4) is very small. Furthermore, Bravo and Luca [20] showed that

$$\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1} \tag{3.6}$$

holds for all $n \geq 1$ and $k \geq 2$.

The following lemma is a simple result, which is a small variation of the right–hand side of previous inequality (3.6) (see [20, Lemma 2]).

Lemma 3.2. *For every positive integer $n \geq 2$, we have*

$$F_n^{(k)} \leq 2^{n-2}.$$

Moreover, if $2 \leq n \leq k+1$, then the equality is fulfilled.

3.2.2 The Pell sequence

An explicit Binet formula for the sequence $P = \{P_n\}_{n\geq 0}$ is well-known and is useful for our purposes. Namely, we have that

$$P_m = \frac{\gamma^m - \delta^m}{2\sqrt{2}} \quad \text{holds for all} \quad m \geq 0, \tag{3.7}$$

where $(\gamma, \delta) = (1+\sqrt{2}, 1-\sqrt{2})$ are the roots of the characteristic polynomial $x^2 - 2x - 1$. In particular, it easily implies that the inequality

$$\gamma^{m-2} \leq P_m \leq \gamma^{m-1} \quad \text{holds for all} \quad m \geq 1. \tag{3.8}$$

Remark 3.1. *We note that for linear recurrence sequences having a dominant root*[1]*, which is a Pisot number, one expects an inequality similar to that of (3.6), or (3.8), that shows the exponential growth of the sequence.*

3.3 The proof of Theorem 3.1

Assume that $(n, m) \neq (n_1, m_1)$ are pairs of positive indices with $n, n_1 \geq 2$ such that

$$F_n^{(k)} - F_{n_1}^{(k)} = P_m - P_{m_1}. \tag{3.9}$$

[1]We say that a linear recurrence sequence has a dominant root if one of the roots of its characteristic polynomial has strictly largest absolute value.

Note that $n \neq n_1$, since otherwise $(n, m) = (n_1, m_1)$. We may assume that $n > n_1$ giving $n \geq 3$. Then both sides of equation (3.9) are positive and therefore $m > m_1$. Thus, $m \geq 2$.

If $k = 2$, then $F^{(k)}$ is the Fibonacci sequence $\{F_n\}_{n\geq 0}$ and in this case we already know the solutions of equation (3.9) as mentioned before. It was proved in [43] that all integers c admitting at least two representations as a difference between a Fibonacci and a Pell number are

$$c \in \{-4, 0, 1, 2, 3, 19\}.$$

All representations, which have been included in the statement of Theorem 3.1, are

$$\begin{array}{rcccccl} -4 & = & F_6 - P_4 & = & F_2 - P_3; & & \\ 0 & = & F_2 - P_1 & = & F_3 - P_2 & = & F_5 - P_3; \\ 1 & = & F_3 - P_1 & = & F_4 - P_2 & = & F_7 - P_4; \\ 2 & = & F_4 - P_1 & = & F_{16} - P_9; & & \\ 3 & = & F_5 - P_2 & = & F_6 - P_3; & & \\ 19 & = & F_{11} - P_6 & = & F_8 - P_2. & & \end{array}$$

From now on, we assume that $k \geq 3$. Note that if $n = 3$, then, by (3.9), we have $n_1 = 2$ giving $(m, m_1) = (2, 1)$, which is the first parametric family of solutions. Hence, we can assume that $n \geq 4$.

3.3.1 The case $4 \leq n \leq k + 1$

In this case, by Lemma 3.2 we have that $F_n^{(k)} = 2^{n-2}$ and so our problem is reduced to finding all integers c having at least two representations of the form

$$2^{n-2} - P_m = c$$

in positive integers n and m with $n \geq 4$ and $m \geq 3$. But this last problem was completely solved by Hernane, Luca, Rihane and Togbé in [42]. What they proved is that the only integers c having at least two representations of the form $P_s - 2^t$ are

$$c \in \{-3, 0, 1, 4\}.$$

In addition to this, they found all the representations of the above integers c as $P_s - 2^t$ with integers $s \geq 1$ and $t \geq 0$. Namely

$$\begin{array}{rcl} -3 & = & P_5 - 2^5 = P_3 - 2^3 = P_1 - 2^2; \\ 0 & = & P_2 - 2^1 = P_1 - 2^0; \\ 1 & = & P_3 - 2^2 = P_2 - 2^0; \\ 4 & = & P_4 - 2^3 = P_3 - 2^0. \end{array}$$

The above solutions give the parametric families of solutions listed in Theorem 3.1. We point out that there is a typo in the statement of the previous result given in [42] corresponding to the representations of $c = -3$. This has been fixed here.

3.3.2 The case $n \geq k + 2$

Here, by using inequalities (3.6), (3.8) and Lemma 3.2, we get

$$\alpha^{n-4} \leq F_{n-2}^{(k)} + \ldots + F_{n-k}^{(k)} = F_n^{(k)} - F_{n-1}^{(k)} \leq F_n^{(k)} - F_{n_1}^{(k)} = P_m - P_{m_1} < P_m \leq \gamma^{m-1},$$

$$2^{n-2} \geq F_n^{(k)} \geq F_n^{(k)} - F_{n_1}^{(k)} = P_m - P_{m_1} > \gamma^{m-3},$$

as well as

$$\alpha^{n-1} > \gamma^{m-3}.$$

We record these inequalities as follows.

Lemma 3.3. *The inequalities*

$$\gamma^{m-1} > \alpha^{n-4}, \qquad 2^{n-2} > \gamma^{m-3} \qquad \textit{and} \qquad \alpha^{n-1} > \gamma^{m-3}$$

hold. In particular, $n \geq m$.

To solve equation (3.9), we need an upper bound for n.

Bounding n

Let us suppose first that $n \leq 300$. In this case, since $n \geq k + 2$, we get $k \leq 298$. We next find the solutions of equation (3.9) in the small range. Indeed, by Lemma 3.3, we can write

$$c(n-4) + 1 < m < c(n-1) + 3$$

where $c = (\log \alpha)/(\log \gamma)$. Now, for each $k \in [3, 298]$ and $n \in [k+2, 300]$, we created the sets

$$\mathcal{F}ib_{k,n} = \{F_n^{(k)} - F_{n_1}^{(k)} : n_1 \in [2, n-1]\}$$

and

$$\mathcal{P}_{k,n} = \{P_m - P_{m_1} : m \in [\lfloor c(n-4) + 1\rceil, \lfloor c(n-1) + 3\rceil], m_1 \in [1, m-1]\}.$$

With the help of *Mathematica*, we found the intersections

$$\mathcal{F}ib_{k,n} \cap \mathcal{P}_{k,n} \quad \text{for all} \quad (k,n) \in [3,298] \times [k+2,300].$$

This search reveals that all indices (k,n) for which $\mathcal{F}ib_{k,n} \cap \mathcal{P}_{k,n}$ is not empty belong to the set

$$\left\{ \begin{array}{llllll} (3,5), & (3,6), & (3,9), & (3,10), & (3,15), & (4,6), \\ (4,7), & (4,8), & (5,7), & (5,11), & (8,10), & (8,12) \end{array} \right\}.$$

These indices give the sporadic solutions given in the statement of Theorem 3.1. Thus, we can assume from now on that $n > 300$. Then, by Lemma 3.3 we get $m > 163$. In the rest of this subsection, we shall work with $\mathbb{K} := \mathbb{Q}(\alpha, \gamma)$ so that $D = [\mathbb{K} : \mathbb{Q}] \leq 2k$. Put

$$C_1 := 1.4 \times 30^6 \times 3^{4.5} \times (2k)^2 (1 + \log 2k)(1 + \log n).$$

Combining (3.7) and (3.9), we have

$$\begin{aligned} \left| f_k(\alpha)\alpha^{n-1} - \frac{\gamma^m}{2\sqrt{2}} \right| &= \left| f_k(\alpha)\alpha^{n-1} - (F_n^{(k)} - F_{n_1}^{(k)}) - \frac{\delta^m}{2\sqrt{2}} - P_{m_1} \right| \\ &\leq \left| f_k(\alpha)\alpha^{n-1} - F_n^{(k)} \right| + \frac{|\delta|^m}{2\sqrt{2}} + F_{n_1}^{(k)} + P_{m_1}. \end{aligned}$$

We now use (3.5), (3.6) and (3.8), together with the fact that $|\delta| < 1$, to obtain

$$\left| f_k(\alpha)\alpha^{n-1} - \frac{\gamma^m}{2\sqrt{2}} \right| < 1 + \alpha^{n_1 - 1} + \gamma^{m_1 - 1}.$$

Dividing both sides by $f_k(\alpha)\alpha^{n-1}$ and using that $f_k(\alpha) > 1/2$, we get

$$\begin{aligned} \left| f_k(\alpha)^{-1}\alpha^{-(n-1)}\gamma^m (2\sqrt{2})^{-1} - 1 \right| &< \frac{2\alpha}{\alpha^n} + \frac{2}{\alpha^{n-n_1}} + \frac{2\gamma^{m_1-1}}{\alpha^{n-1}} \\ &< \frac{2\alpha}{\alpha^n} + \frac{2}{\alpha^{n-n_1}} + \frac{2\gamma^2}{\alpha^{m-m_1}} \end{aligned}$$

which implies

$$\left| f_k(\alpha)^{-1}\alpha^{-(n-1)}\gamma^m (2\sqrt{2})^{-1} - 1 \right| < \frac{18}{\alpha^{\min\{n-n_1, m-m_1\}}}. \tag{3.10}$$

In order to obtain absolute upper bounds for k and n, we need to apply several times Matveev's theorem, and to do this we must ensure that the corresponding linear forms do not vanish. This task will be done later for all the linear forms treated in this paper.

Let $\Lambda_1 := f_k(\alpha)^{-1}\alpha^{-(n-1)}\gamma^m (2\sqrt{2})^{-1} - 1$. We first apply Matveev's theorem to the left–hand side of (3.10) with the parameters $t = 3$ and

$$\gamma_1 := 2\sqrt{2} f_k(\alpha), \quad \gamma_2 := \alpha, \quad \gamma_3 := \gamma,$$

$$b_1 := -1, \quad b_2 := -(n-1), \quad b_3 := m.$$

Notice that $\mathbb{K}$ contains $\gamma_1, \gamma_2, \gamma_3$. By Lemma 3.3 we have $m \leq n$ and so we take $B := n$. From the properties of the logarithmic height function, we have that

$$\begin{aligned} h(\gamma_1) = h(2\sqrt{2}f_k(\alpha)) &\leq h(2\sqrt{2}) + h(f_k(\alpha)) \\ &\leq \frac{3}{2}\log 2 + 2\log k \leq 3\log k \end{aligned}$$

for all $k \geq 3$, where we used the estimate $h(f_k(\alpha)) < 2\log k$ (see [16, p. 111]). Hence, we can take $A_1 := 6k\log k$. Furthermore, we can take $A_2 := 2\log 2$ and $A_3 := k\log\gamma$. Applying Matveev's theorem, we deduce that

$$\log|\Lambda_1| > -C_1A_1A_2A_3 > -2.52 \times 10^{13}k^4\log^2 k\log n$$

which together with (3.10) implies

$$\min\{n - n_1, m - m_1\} < 5.24 \times 10^{13}k^4\log^2 k\log n. \tag{3.11}$$

Now, put

$$\mathcal{T} := \{n - n_1, m - m_1\} = \{t_1, t_2\},$$

where $t_1 \leq t_2$. We shall prove the following lemma.

Lemma 3.4. *If (n, n_1, m, m_1) is a solution of (3.9) with $n > 300$, $n \geq k+2$ and $n > n_1$, then*

$$\max\{n - n_1, m - m_1\} < 4.06 \times 10^{26}k^8\log^3 k\log^2 n.$$

Proof. We want to apply Matveev's theorem to the linear forms

$$\Lambda_{\{2,1\}} := 2\sqrt{2}f_k(\alpha)(1 - \alpha^{n_1-n})\alpha^{n-1}\gamma^{-m} - 1$$

and

$$\Lambda_{\{2,2\}} := \frac{1 - \gamma^{m_1-m}}{2\sqrt{2}f_k(\alpha)}\alpha^{-(n-1)}\gamma^m - 1.$$

In order to deduce upper bounds on t_i, we suppose $\min\mathcal{T} = n - n_1$ and use (3.9) to obtain

$$\begin{aligned} \left|f_k(\alpha)(\alpha^{n-1} - \alpha^{n_1-1}) - \frac{\gamma^m}{2\sqrt{2}}\right| &\leq \left|f_k(\alpha)(\alpha^{n-1} - \alpha^{n_1-1}) - F_n^{(k)} + F_{n_1}^{(k)}\right| \\ &+ \frac{|\delta|^m}{2\sqrt{2}} + P_{m_1} < \frac{3}{2} + \gamma^{m_1-1}. \end{aligned}$$

Dividing through by $\gamma^m/(2\sqrt{2})$, we get,

$$\begin{aligned}\left|2\sqrt{2}f_k(\alpha)(1-\alpha^{n_1-n})\alpha^{n-1}\gamma^{-m}-1\right| &< \frac{3\sqrt{2}}{\gamma^m}+\frac{2\sqrt{2}\gamma^{m_1-1}}{\gamma^m}\\ &< \frac{4\sqrt{2}}{\gamma^{m-m_1}} < \frac{8}{\alpha^{m-m_1}}\cdot \end{aligned} \tag{3.12}$$

On the other hand, assuming that $\min\mathcal{T}=m-m_1$ and using (3.9) once again, we have

$$\begin{aligned}\left|f_k(\alpha)\alpha^{n-1}-\frac{\gamma^m}{2\sqrt{2}}+\frac{\gamma^{m_1}}{2\sqrt{2}}\right| &= \left|f_k(\alpha)\alpha^{n-1}-(F_n^{(k)}-F_{n_1}^{(k)})-\frac{\delta^m}{2\sqrt{2}}+\frac{\delta^{m_1}}{2\sqrt{2}}\right|\\ &< \frac{3}{2}+\alpha^{n_1-1}.\end{aligned}$$

Thus,

$$\left|\frac{(1-\gamma^{m_1-m})}{2\sqrt{2}f_k(\alpha)}\alpha^{-(n-1)}\gamma^m-1\right| < \frac{3\alpha}{\alpha^n}+\frac{2}{\alpha^{n-n_1}} < \frac{8}{\alpha^{n-n_1}}\cdot \tag{3.13}$$

Let us see how to deduce the upper bounds on t_i. Note that

$$\Lambda_{\{2,1\}} := \gamma_4^{b_4}\gamma_2^{b_2}\gamma_3^{-b_3}-1 \quad \text{and} \quad \Lambda_{\{2,2\}} := \gamma_5^{b_5}\gamma_2^{-b_2}\gamma_3^{b_3}-1$$

where

$$\gamma_2 := \alpha, \quad \gamma_3 := \gamma, \quad \gamma_4 := 2\sqrt{2}f_k(\alpha)(1-\alpha^{n_1-n}), \quad \gamma_5 := \frac{1-\gamma^{m_1-m}}{2\sqrt{2}f_k(\alpha)},$$

and

$$b_2 := n-1, \quad b_3 := m, \quad b_4 := b_5 = 1.$$

In order to get lower bounds on $\Lambda_{\{2,1\}}$ and $\Lambda_{\{2,2\}}$, one can use the properties of the logarithmic height to get,

$$h(\gamma_i) \leq \begin{cases} \frac{3}{2}\log 2+2\log k+(n-n_1)\frac{\log 2}{k}+\log 2, & if \quad i=4;\\ \frac{3}{2}\log 2+2\log k+(m-m_1)\frac{\log\gamma}{2}+\log 2, & if \quad i=5.\end{cases}$$

Since $(\log 2)/k < (\log\gamma)/2$, it follows from (3.11) that

$$h(\gamma_i) < 2.32\times 10^{13}k^4\log n\log^2 k.$$

Moreover, we take

$$A_4 = A_5 := 4.64\times 10^{13}k^5\log n\log^2 k.$$

Furthermore, we can take $A_2 := 2\log 2$ and $A_3 := k\log\gamma$ as before. Finally, since $\max\{n-1, m, 1\} \le n$, we can take $B := n$. We then get that

$$\log|\Lambda_{\{i,j\}}| > -C_1 A_2 A_3 A_4 \ge -1.95\times 10^{26} k^8 \log^2 n \log^3 k$$

for $\{i,j\} \in \{\{2,1\},\{2,2\}\}$. Comparing this with (3.12) and (3.13), we get

$$\max\{n-n_1, m-m_1\} < 4.06\times 10^{26} k^8 \log^3 k \log^2 n,$$

which proves the lemma. □

One more time; again, we must apply Matveev's theorem. This time we use equation (3.9) to obtain

$$\begin{aligned} \left| f_k(\alpha)(\alpha^{n-1} - \alpha^{n_1-1}) - \frac{\gamma^m - \gamma^{m_1}}{2\sqrt{2}} \right| \quad &\le \quad \left| f_k(\alpha)(\alpha^{n-1} - \alpha^{n_1-1}) - F_n^{(k)} + F_{n_1}^{(k)} \right| \\ &+ \quad \frac{|\delta^m - \delta^{m_1}|}{2\sqrt{2}} < 2, \end{aligned}$$

leading to

$$\begin{aligned} \left| \frac{1-\gamma^{m_1-m}}{2\sqrt{2} f_k(\alpha)(1-\alpha^{n_1-n})} \alpha^{-(n-1)} \gamma^m - 1 \right| &< \frac{2}{f_k(\alpha)\alpha^{n-1}(1-\alpha^{n_1-n})} \\ &< \frac{8}{\alpha^n(1-\alpha^{n_1-n})} < \frac{24}{\alpha^n}. \end{aligned} \tag{3.14}$$

We now write

$$\Lambda_3 := \frac{1-\gamma^{m_1-m}}{2\sqrt{2} f_k(\alpha)(1-\alpha^{n_1-n})} \alpha^{-(n-1)} \gamma^m - 1 = \gamma_6^{b_6} \gamma_2^{b_2} \gamma_3^{b_3} - 1,$$

where

$$\gamma_6 := \frac{1-\gamma^{m_1-m}}{2\sqrt{2} f_k(\alpha)(1-\alpha^{n_1-n})}, \quad b_6 := 1$$

and $\gamma_2, \gamma_3, b_2, b_3$ are given as before. From the properties of the logarithmic height function, we have that

$$\begin{aligned} h(\gamma_6) &\le \frac{3}{2}\log 2 + 2\log k + (m-m_1)\frac{\log\gamma}{2} + \log 2 + (n-n_1)\frac{\log 2}{k} + \log 2 \\ &< 3.19\times 10^{26} k^8 \log^3 k \log^2 n. \end{aligned}$$

So we can take $A_6 := 6.38\times 10^{26} k^9 \log^3 k \log^2 n$. We also take $B := n$. It then follows from Matveev's theorem applied to (3.14), after some calculations, that

$$n < 5.57\times 10^{39} k^{12} \log^4 k \log^3 n,$$

which is equivalent to

$$\frac{n}{\log^3 n} < 5.57 \times 10^{39} k^{12} \log^4 k.$$

Then, from Lemma 3.1, and after some elementary algebra, we have

$$n < 4.46 \times 10^{46} k^{12} \log^7 k. \tag{3.15}$$

The case of large k

Here we shall work with $\mathbb{K} := \mathbb{Q}(\gamma)$ which has $D = [\mathbb{K} : \mathbb{Q}] = 2$. We take

$$C_2 := 1.4 \times 30^6 \times 3^{4.5} \times 2^2 (1 + \log 2)(1 + \log n).$$

From now on, we assume that $k > 782$. For such k we have

$$n < 4.46 \times 10^{46} k^{12} \log^7 k < 2^{k/2}.$$

At this point, we require an important estimate due to Bravo, Gómez and Luca [18]. They proved that if $n < 2^{k/2}$, then the *nth* k-Fibonacci number can be written as

$$F_n^{(k)} = 2^{n-2}(1 + \zeta) \quad \text{where} \quad |\zeta| < 1/2^{k/2}. \tag{3.16}$$

Then, by using the above estimate (3.16) and (3.9), we get

$$\begin{aligned} \left|2^{n-2} - \frac{\gamma^m}{2\sqrt{2}}\right| &= \left|\left(2^{n-2} - F_n^{(k)}\right) + \left(F_n^{(k)} - \frac{\gamma^m}{2\sqrt{2}}\right)\right| \\ &< \frac{2^{n-2}}{2^{k/2}} + 1 + \alpha^{n_1 - 1} + \gamma^{m_1 - 1}. \end{aligned}$$

So, by Lemma 3.3, we obtain

$$\begin{aligned} \left|(\sqrt{2})^{-2n+1}\gamma^m - 1\right| &< \frac{1}{2^{k/2}} + \frac{4}{2^n} + \frac{2}{2^{n-n_1}} + \frac{\gamma^{m_1-1}}{2^{n-2}} \\ &< \frac{1}{2^{k/2}} + \frac{4}{2^n} + \frac{2}{2^{n-n_1}} + \frac{\gamma^2}{2^{m-m_1}} \\ &< \frac{13}{2^{\min\{k/2, n-n_1, m-m_1\}}}. \end{aligned} \tag{3.17}$$

Let Λ_4 be the expression inside the absolute value on the left–hand side of (3.17). We apply again Matveev's theorem with the data

$$\gamma_1 := \sqrt{2}, \quad \gamma_2 := \gamma, \quad b_1 := -2n + 1, \quad b_2 := m.$$

We begin by noticing that the two numbers γ_1, γ_2 are positive real numbers and belong to the field $\mathbb{K}$. It follows that we can take $B := 2n$, $A_1 := \log 2$ and $A_2 := \log \gamma$. Then, the left–hand side of (3.17) is bounded below by

$$\begin{aligned} \log|\Lambda_4| &> -1.4 \times 30^5 \times 2^{4.5} \times 2^2(1+\log 2)(1+\log(2n))(\log 2)(\log\gamma) \\ &> -2 \times 10^{10}\log n. \end{aligned}$$

Comparing this with (3.17), we get

$$\min\{k/2, n-n_1, m-m_1\} < 2.9 \times 10^{10}\log n. \tag{3.18}$$

Now the argument is split into three cases.

Case 1. $\min\{k/2, n-n_1, m-m_1\} = k/2$. Here, by inequalities (3.15) and (3.18), we obtain

$$\begin{aligned} \frac{k}{2} < 2.9 \times 10^{10}\log n &< 2.9 \times 10^{10}\log(4.46 \times 10^{46} \times k^{12} \times \log^7 k) \\ &< 4 \times 10^{12}\log k, \end{aligned}$$

which implies $k < 10^{15}$.

Case 2. $\min\{k/2, n-n_1, m-m_1\} = n-n_1$. Using (3.9) and (3.16) as well as the right–hand side of (3.8), we get

$$\begin{aligned} \left|2^{n-2} - 2^{n_1-2} - \frac{\gamma^m}{2\sqrt{2}}\right| &\le \left|2^{n-2} - 2^{n_1-2} - F_n^{(k)} + F_{n_1}^{(k)}\right| + \left|F_n^{(k)} - F_{n_1}^{(k)} - \frac{\gamma^m}{2\sqrt{2}}\right| \\ &< \frac{2^{n-2}}{2^{k/2}} + \frac{2^{n_1-2}}{2^{k/2}} + \frac{\delta^m}{2\sqrt{2}} + \gamma^{m_1-1} \\ &< \frac{2^{n-1}}{2^{k/2}} + 1 + \gamma^{m_1-1}. \end{aligned}$$

We now divide through both sides by $2^{n-2}(1-2^{n_1-n})$ and use Lemma 3.3 to get

$$\begin{aligned} \left|\frac{\gamma^m}{2\sqrt{2}}(2^{n-2}(1-2^{n_1-n}))^{-1} - 1\right| &< \frac{4}{2^{k/2}} + \frac{8}{2^n} + \frac{2\gamma^{m_1-1}}{2^{n-2}} \\ &< \frac{4}{2^{k/2}} + \frac{8}{2^n} + \frac{2\gamma^2}{2^{m-m_1}}. \end{aligned}$$

In the above we have also used the fact that $1 - 2^{n_1-n} \ge 1/2$ because $n - n_1 \ge 1$. On the other hand, since $n \ge k+2$ we obtain $1/2^n < 1/2^{k/2}$ and so

$$\begin{aligned} \left|\frac{\gamma^m}{\sqrt{2}}2^{-(n-1)}(1-2^{n_1-n})^{-1} - 1\right| &< \frac{4}{2^{k/2}} + \frac{8}{2^{k/2}} + \frac{2\gamma^2}{2^{m-m_1}} \\ &< \frac{24}{2^{\min\{k/2, m-m_1\}}}. \end{aligned} \tag{3.19}$$

Let $\Lambda_5 := \gamma_1^{b_1}\gamma_2^{b_2}\gamma_3^{b_3} - 1$ where $\gamma_1 := \sqrt{2}(1-2^{n_1-n})$, $\gamma_2 := 2$ and $\gamma_3 := \gamma$. Put also $b_1 := -1$, $b_2 := -(n-1)$ and $b_3 := m$. Notice that $\mathbb{K}$ contains $\gamma_1, \gamma_2, \gamma_3$. Further, we may take $B := n$. From the properties of the logarithmic height function and (3.18), we have that

$$h(\gamma_1) \le \frac{1}{2}\log 2 + (n-n_1)\log 2 < 2.1\times 10^{10}\log n.$$

So, we take $A_1 := 4.2\times 10^{10}\log n$, $A_2 := 2\log 2$, $A_3 := \log\gamma$. By Matveev's theorem, we deduce that

$$\log|\Lambda_5| > -1.1\times 10^{23}\log n$$

and comparing it with (3.19), we obtain

$$\min\{k/2, m-m_1\} < 1.6\times 10^{23}\log^2 n.$$

If $\min\{k/2, m-m_1\} = k/2$, then we proceed as in Case 1 obtaining $k < 10^{28}$. Otherwise, we have

$$m - m_1 < 1.6\times 10^{23}\log^2 n.$$

Case 3. $\min\{k/2, n-n_1, m-m_1\} = m-m_1$. Here, applying (3.16) and using once again (3.9) we deduce

$$\left|2^{n-2} - \frac{\gamma^m}{2\sqrt{2}} + \frac{\gamma^{m_1}}{2\sqrt{2}}\right| < \frac{2^{n-2}}{2^{k/2}} + 2^{n_1-1} + 1.$$

Dividing by 2^{n-2} and rearranging the resulting inequality, we get

$$\left|2^{-(n-2)}\gamma^m(2\sqrt{2})^{-1}(1-\gamma^{m_1-m}) - 1\right| < \frac{7}{2^{\min\{k/2, n-n_1\}}}. \tag{3.20}$$

We now proceed as in the previous case to obtain

$$\min\{k/2, n-n_1\} < 1.8\times 10^{23}\log^2 n,$$

and so $k \le 10^{28}$ or $n - n_1 < 1.8\times 10^{23}\log^2 n$. In any case we obtain that

$$\max\{n-n_1, m-m_1\} < 1.8\times 10^{23}\log^2 n \quad \text{or} \quad k < 10^{28}.$$

Returning to the equation (3.9) and using once again (3.16), one gets

$$\left|2^{n-2} - 2^{n_1-2} - \frac{\gamma^m}{2\sqrt{2}} + \frac{\gamma^m}{2\sqrt{2}}\right| < \frac{2^{n-1}}{2^{k/2}} + 1,$$

and dividing it by $2^{n-2}(1-2^{n_1-n})$, we arrive at

$$\begin{aligned}\left|\gamma^m(1-\gamma^{m_1-m})(\sqrt{2})^{-1}2^{-(n-1)}(1-2^{n_1-n})^{-1} - 1\right| &< \frac{4}{2^{k/2}} + \frac{8}{2^n}\\ &< \frac{12}{2^{k/2}}.\end{aligned} \tag{3.21}$$

Here, we put

$$\Lambda_6 := \gamma^m(1-\gamma^{m_1-m})(\sqrt{2})^{-1}2^{-(n-1)}(1-2^{n_1-n})^{-1} - 1.$$

As before we take $\mathbb{K} := \mathbb{Q}(\sqrt{2})$ and the parameters

$$\begin{array}{lll} \gamma_6 := \dfrac{1-\gamma^{m_1-m}}{\sqrt{2}(1-2^{n_1-n})}, & \gamma_2 := 2, & \gamma_3 := \gamma, \\ b_1 := 1, & b_2 := -(n-1), & b_3 := m. \end{array}$$

From the properties of the logarithmic height and similar computations done before one can obtain that

$$h(\gamma_6) = h\left(\frac{1-\gamma^{m_1-m}}{\sqrt{2}(1-2^{n_1-n})}\right) < 2.1\times 10^{23}\log^2 n.$$

Now, a new application of Matveev's theorem yields

$$\log|\Lambda_6| > 10^{36}\log^3 n,$$

which combined with (3.21) implies

$$\frac{k}{2}\log 2 < \log 12 + 10^{36}\log^3 n.$$

This, together with (3.15), gives us

$$k < 10^{47} \quad \text{and hence} \quad m \le n < 7.76\times 10^{624}. \tag{3.22}$$

Justifying that $\Lambda_U \neq 0$

We now justify that the linear forms Λ_U, where

$$U \in \{1, \{2,1\}, \{2,2\}, 3, 4, 5, 6\}$$

are nonzero. To do this, let $\mathbb{L} = \mathbb{Q}(\alpha_1,\ldots,\alpha_k,\gamma)$ and let $\sigma_1,\ldots,\sigma_k$ be elements of $Gal(\mathbb{L}/\mathbb{Q})$ such that $\sigma_i(\alpha) = \alpha_i$. Since $k \ge 3$ and γ has degree 2, there exist $i \neq j$ in $\{1,\ldots,k\}$ such that $\sigma_i(\gamma) = \sigma_j(\gamma)$. We now consider the automorphism $\sigma = \sigma_j^{-1}\sigma_i$ and observe that $\sigma(\gamma) = \gamma$, $\sigma(\sqrt{2}) = \sqrt{2}$ and $\sigma(\alpha) \neq \alpha$ because $\sigma_j(\alpha) = \alpha_j \neq \alpha_i$. We will use this fact to show that Λ_U is not zero.

For $U = 1$, the form Λ_1 appears in the left–hand side of (3.10). If Λ_1 were zero, then

$$2\sqrt{2}f_k(\alpha)\alpha^{n-1} = \gamma^m. \tag{3.23}$$

Applying σ to the relation (3.23) and taking absolute values, we get

$$2\sqrt{2}\,|f_k(\alpha_s)|\,|\alpha_s|^{n-1} = \gamma^m$$

where $s \neq 1$ is such that $\sigma(\alpha) = \alpha_s$. But the above equality is impossible since its left–hand side is $< 2\sqrt{2} < 3$ because $|\alpha_s| < 1$ and $|f_k(\alpha_s)| < 1$, whereas its right–hand side is $\geq \gamma^{163} > 10^{62}$. Thus, $\Lambda_1 \neq 0$.

For $U = \{2, 1\}$, the form is the one appearing in the left–hand side of (3.12). If this were zero, then we would get

$$2\sqrt{2} f_k(\alpha)(\alpha^{n-1} - \alpha^{n_1-1}) = \gamma^m.$$

Conjugating this last equation with the automorphism σ used before, and then taking absolute values, we arrive at the equality

$$2\sqrt{2}|f_k(\alpha_s)||\alpha_s^{n-1} - \alpha_s^{n_1-1}| = \gamma^m,$$

for some $s \geq 2$. But this cannot hold because its left–hand side is $< 4\sqrt{2} < 6$, while its right–hand side is $> 10^{62}$ as mentioned before.

For $U = \{2, 2\}$, the form appears in the left–hand side of (3.13). If it is zero, then

$$f_k(\alpha)\alpha^{n-1} = \frac{\gamma^m - \gamma^{m_1}}{2\sqrt{2}},$$

and the same argument as above then shows that

$$|f_k(\alpha_s)||\alpha_s|^{n-1} = \frac{\gamma^m - \gamma^{m_1}}{2\sqrt{2}}.$$

But the last equality is impossible since its left–hand side is < 1, whereas its right–hand side is $\geq \gamma^{m-1}(\gamma - 1)/(2\sqrt{2}) = \gamma^{m-1}/2 \geq \gamma^{162}/2$ for all $m \geq 163$. When $U = 3$, the form appears in the left–hand side of (3.14). Here, repeating the previous argument one can prove that $\Lambda_3 \neq 0$.

For $U \in \{4, 5\}$, the forms appear in the left–hand sides of (3.17) and (3.19), respectively. If these forms were zero, then we would have that

$$\gamma^{2m} = 2^{2n-1} \quad \text{and} \quad \gamma^{2m} = 2(2^{n-1} - 2^{n_1-1})^2,$$

respectively. But the above relations are not possible because no positive power of γ is an integer. Thus, $\Lambda_U \neq 0$ for $U \in \{4, 5\}$. Finally, for $U = 6$, the form appears in the left–hand side of (3.21). If it were zero, we would get the relation

$$\gamma^m - \gamma^{m_1} = \sqrt{2}(2^{n-1} - 2^{n_1-1}). \tag{3.24}$$

Conjugating the above relation in $\mathbb{Q}(\sqrt{2})$, we obtain

$$\delta^m - \delta^{m_1} = -\sqrt{2}(2^{n-1} - 2^{n_1-1}). \tag{3.25}$$

Combining (3.24) and (3.25), we get

$$\sqrt{2}\gamma^{m-1} \leq \gamma^m - \gamma^{m_1} = |\delta^m - \delta^{m_1}| \leq |\delta|^m + |\delta|^{m_1} < 1,$$

which is impossible for $m > 163$. Hence, $\Lambda_6 \neq 0$.

3.3.3 Reducing the bound on k and n

The case of large k

Suppose $k > 782$. During the course of our calculations we got $k < 10^{47}$. We note that the upper bound given for k is too large to find the solutions of equation (3.9) by using a computer, so we make use of Lemma 2.2 several times to reduce it. First, we work with inequality (3.17). To do this, we let

$$z := m \log\gamma - n\log 2 + \frac{1}{2}\log 2.$$

Then (3.17) can be rewritten as

$$|e^z - 1| < \frac{13}{2^{\min\{k/2,\, n-n_1,\, m-m_1\}}}. \tag{3.26}$$

Note that $z \neq 0$ since $\Lambda_4 \neq 0$, so we distinguish the following cases. If $z > 0$, then $e^z - 1 > 0$ and therefore

$$0 < z < \frac{13}{2^{\min\{k/2,\, n-n_1,\, m-m_1\}}},$$

where we used the fact that $x < e^x - 1$ for all $x \neq 0$. Now, if we assume that $z < 0$ and $\min\{k/2,\, n-n_1,\, m-m_1\} \geq 5$, then the right–hand side of (3.26) is $< 1/2$ and so $|e^z - 1| < 1/2$. Then, we get $1 - e^z < 1/2$ implying $e^{-z} = e^{|z|} < 2$. Since $z < 0$, we have

$$0 < |z| < e^{|z|} - 1 = e^{|z|}|e^z - 1| < \frac{26}{2^{\min\{k/2,\, n-n_1,\, m-m_1\}}}.$$

In any case we get that

$$|z| < \frac{26}{2^{\min\{k/2,\, n-n_1,\, m-m_1\}}}$$

holds provided $\min\{k/2,\, n-n_1,\, m-m_1\} \geq 5$. Hence

$$|2m\log\gamma - (2n-1)\log 2| < \frac{52}{2^{\min\{k/2,\, n-n_1,\, m-m_1\}}}.$$

Dividing both sides of the above inequality by $2m\log 2$ we get

$$\left|\frac{\log\gamma}{\log 2} - \frac{2n-1}{2m}\right| < \frac{76}{(2m)2^{\min\{k/2, n-n_1,, m-m_1\}}}.$$

Suppose that $\min\{k/2, n-n_1, m-m_1\} > 2245$. Then the right–hand side above is smaller than $1/(2(2m)^2)$. Indeed, the inequality

$$\frac{76}{(2m)2^{\min\{k/2, n-n_1, m-m_1\}}} < \frac{1}{2(2m)^2},$$

is equivalent to

$$152(2m) < 2^{\min\{k/2, n-n_1, m-m_1\}}$$

and this last inequality is fulfilled because $2m \leq 2n < 2(7.76 \times 10^{624})$ and it has been assumed that $\min\{k/2, n-n_1,, m-m_1\} > 2245$. Thus, for

$$\min\{k/2, n-n_1, m-m_1\} > 2245,$$

we have

$$\left|\frac{\log\gamma}{\log 2} - \frac{2n-1}{2m}\right| < \frac{1}{2(2m)^2}.$$

By Lemma 2.1 (see (2.12)), $(2n-1)/(2m) = p_s/q_s$ for some convergent p_s/q_s of

$$\frac{\log\gamma}{\log 2} = [a_0, a_1, a_2, \ldots] = [1, 3, 1, 2, 6, 1, 2, 11, 2, \ldots].$$

Since $q_{1222} \leq 2m < q_{1223}$, it follows that $s \leq 1222$, and

$$a_{s+1} \leq \max\{a_j : 0 \leq j \leq 1223\} = 2030.$$

By Lemma 2.1 (see (2.13)), we get that

$$\frac{1}{2032(2m)^2} < \left|\frac{\log\gamma}{\log 2} - \frac{2n-1}{2m}\right| < \frac{76}{(2m)2^{\min\{k/2, n-n_1,, m-m_1\}}},$$

giving

$$2^{\min\{k/2, n-n_1, m-m_1\}} < 308864m < 2.4 \times 10^{630}.$$

Thus $\min\{k/2, n-n_1, m-m_1\} < 2095$, which is a contradiction. Consequently,

$$\min\{k/2, n-n_1, m-m_1\} \leq 2245. \tag{3.27}$$

If $\min\{k/2, n-n_1, m-m_1\} = m-m_1$, we then go back to inequality (3.20), and defining

$$\Theta := \frac{1}{2}(2m\log\gamma - (2n-1)\log 2 + 2\log(1-\gamma^{m_1-m})),$$

we obtain

$$0 < |\Theta| < \frac{14}{2^{\min\{k/2,\, n-n_1\}}} \tag{3.28}$$

provided $\min\{k/2, n-n_1\} \geq 4$. Multiplying both sides of the above inequality by $2/\log 2$ and putting $t = m - m_1$, we get

$$0 < |2m\tau - (2n-1) + \mu_t| < \frac{A}{B^{\min\{k/2,\, n-n_1\}}}, \tag{3.29}$$

where

$$\tau := \frac{\log\gamma}{\log 2}, \quad A := 41, \quad B = 2 \quad \text{and} \quad \mu_t := \frac{2\log(1-\gamma^{-t})}{\log 2}.$$

Clearly τ is an irrational number. We put $M := 1.552 \times 10^{625}$ which is an upper bound for $2m$ according to (3.22). It then follows from Lemma 2.2, applied to inequality (3.29), that

$$\min\{k/2, n-n_1\} < \frac{\log(Aq/\varepsilon)}{\log B},$$

where $q > 6M$ is a denominator of a convergent of the continued fraction of τ such that $\varepsilon = ||\mu q|| - M||\tau q|| > 0$. A computer search with *Mathematica* revealed that $\log(Aq/\varepsilon)/\log B \leq 2860$ for all choices $t \in \{1, 2, \ldots, 2245\}$ except when $t = 1, 4$. Thus,

$$\min\{k/2,\ n-n_1\} < 2860 \quad \text{for all} \quad t \in \{1, 2, \ldots, 2245\}, \quad t \neq 1, 4. \tag{3.30}$$

We could not study the cases $t = 1, 4$ as before because when applying Lemma 2.2 to the expression (3.29), the corresponding parameter μ appearing in Lemma 2.2 is an integer linear combination of 1 and τ. For these special cases we have that

$$\Theta := (m-1)\log\gamma - (n-1)\log 2 \quad \text{or} \quad \Theta := (m-2)\log\gamma - (n-3)\log 2,$$

depending on whether $t = 1$ or 4, respectively. By (3.28), we get

$$\left|\frac{\log\gamma}{\log 2} - \frac{x}{y}\right| < \frac{21}{y \cdot 2^{\min\{k/2, n-n_1\}}},$$

where $(x, y) = (n-1, m-1)$ if $t = 1$ or $(x, y) = (n-3, m-2)$ if $t = 4$. By the same arguments used for proving (3.27) one gets in both cases

$$\min\{k/2, n-n_1\} \leq 2252. \tag{3.31}$$

If $\min\{k/2,\, n-n_1, m-m_1\} = n-n_1$, then we work with (3.19) and apply Lemma 2.2 to obtain

$$\min\{k/2,\, m-m_1\} < 2100.$$

Based on the previous analysis, we can conclude that

$$\max\{n-n_1, m-m_1\} < 2860.$$

Returning to (3.21) and arguing as in the proof of (3.29) we conclude that the inequality

$$\left|2m\left(\frac{\log\gamma}{\log 2}\right) - (2n-1) + \frac{2\log\left((1-\gamma^{m_1-m})/(1-2^{n_1-n})\right)}{\log 2}\right| < \frac{70}{2^{k/2}} \tag{3.32}$$

holds for all $k \geq 10$. We next apply Lemma 2.2 to inequality (3.32) for all the choices

$$n-n_1, m-m_1 \in \{1, 2, \ldots, 2860\}$$

except for the cases $(n-n_1, m-m_1) = (1,1), (1,4)$. Here, we got $k < 5740$. In these last cases, from (3.32) we get that

$$\left|\frac{\log\gamma}{\log 2} - \frac{n-2}{m-1}\right| < \frac{35}{(m-1)2^{k/2}} \quad \text{and} \quad \left|\frac{\log\gamma}{\log 2} - \frac{n-4}{m-2}\right| < \frac{35}{(m-2)2^{k/2}},$$

depending on whether $(n-n_1, m-m_1) = (1,1)$ or $(1,4)$, respectively. In both cases we apply Legendre's result obtaining $k < 4504$. In conclusion, we get that $k < 5750$ always holds. This leads to a substantial reduction of the upper bound on n, namely

$$m \leq n < 2.1 \times 10^{98}.$$

With this new upper bounds for k and n, we repeat all the process to finally get $k \leq 782$, which is a contradiction.

The case of small k

Suppose now that $k \in [3, 782]$. In order to apply Lemma 2.2 once again, we put

$$\Gamma := m\log\gamma - n\log\alpha + \log(\alpha/(2\sqrt{2}f_k(\alpha))).$$

For technical reasons we assume that $\min\{n-n_1, m-m_1\} \geq 7$. It then follows from (3.10) that

$$\left|m\left(\frac{\log\gamma}{\log\alpha}\right) - (n-1) + \frac{\log(1/(2\sqrt{2}f_k(\alpha)))}{\log\alpha}\right| < \frac{75}{\alpha^{\min\{n-n_1, m-m_1\}}}.$$

Applying Lemma 2.2 in the above inequality for all $k \in [3, 782]$, we obtain that

$$\min\{n - n_1, m - m_1\} \leq 308.$$

Again we distinguish two cases depending on $\min\{n-n_1, m-m_1\}$. At this point, we work with inequalities (3.12) or (3.13) depending on whether $\min\{n - n_1, m - m_1\} = n - n_1$ or $m - m_1$, respectively, and then we apply Lemma 2.2 as before to obtain that

$$\max\{n - n_1, m - m_1\} \leq 618.$$

Finally, we go back to (3.14) and put

$$\lambda := m \log \gamma - (n-1) \log \alpha + \log \left(\frac{1 - \gamma^{m_1 - m}}{\sqrt{2} f_k(\alpha)(1 - \alpha^{n_1 - n})} \right).$$

Assuming $n \geq 7$ and setting $t := m - m_1$, $\ell := n - n_1$, it follows from (3.14) that

$$0 < |m\tau - n + \mu_{t,\ell}| < \frac{A}{B^n}, \tag{3.33}$$

where $A := 100$, $B := \alpha$ and

$$\tau := \frac{\log \gamma}{\log \alpha}, \quad \mu_{t,\ell} := \frac{\log\left((1 - \gamma^{-t})/(\sqrt{2} f_k(\alpha)(1 - \alpha^{-\ell}))\right)}{\log \alpha}.$$

We now apply Lemma 2.2 to inequality (3.33) for all the choices $t, \ell \in [1, 618]$ in order to get a small absolute upper bound for n. Indeed, with the help of *Mathematica* we found that $n < 300$, which contradicts our assumption that $n > 300$. This completes the proof of Theorem 3.1.

Chapter 4

Ratios of sums of two Fibonacci numbers equal to powers of 2

In this chapter, we find all solutions to the Diophantine equation $F_n + F_m = 2^a(F_r + F_s)$, where $\{F_k\}_{k\geq 0}$ is the Fibonacci sequence. This paper continues and extends a previous work which investigated the powers of 2 which are sums of two Fibonacci numbers.

4.1 Introduction

Let $\{F_k\}_{k\geq 0}$ be the *Fibonacci sequence* given by $F_{k+2} = F_{k+1} + F_k$, for all $k \geq 0$, where $F_0 = 0$ and $F_1 = 1$. The problem of determining all integer solutions to Diophantine equations with Fibonacci numbers has gained a considerable amount of interest among the mathematicians and there is a very broad literature on this subject. Also, there is the Lucas sequence, which is as important as the Fibonacci sequence. The Lucas sequence $\{L_k\}_{k\geq 0}$ follows the same recursive pattern as the Fibonacci numbers, but with initial conditions $L_0 = 2$ and $L_1 = 1$. For the beauty and rich applications of these numbers and their relatives one can see Koshy's book [50].

This research continues and extends the previous work [22] which investigated the powers of 2 which are sums of two Fibonacci numbers. To be more precise, we find all solutions of the Diophantine equation

$$F_n + F_m = 2^a(F_r + F_s) \tag{4.1}$$

in non negative integers n, m, a, r and s. First of all, we introduce some terminology. Given a positive integer N the Zeckendorf decomposition of N is a representation of the form

$$N = F_{n_1} + F_{n_2} + \cdots + F_{n_k},$$

where $n_i - n_{i+1} \geq 2$. This always exists and up to identifying F_1 with F_2, it is unique. In (4.1), we ignore the solutions for which $n = m = r = s = 0$ (and any $a \geq 0$). If one or more of the Fibonacci numbers involved in (4.1) equals 1, we then assume that its index is 2. Finally, when $N = F_n + F_m$ and $M = F_r + F_s$, we assume that $n > m \geq 0,\ r > s \geq 0$ and that the above representations are the Zeckendorf decompositions of N and M, respectively. This rules out cases like $m = n - 1$ for which $N = F_n + F_{n-1} = F_{n+1}$, as well as $n = m$ for which $N = F_n + F_n = 2F_n = F_{n+1} + F_{n-2}$. Finally, we also ignore the trivial diagonal solutions $(n, m) = (r, s)$ and $a = 0$. The rest of solutions will be called *non-degenerate*.

The theorem is the following.

Theorem 4.1. *Equation (4.1) has two parametric families of non-degenerate solutions (n, m, a, r, s) with $n > m \geq 0$ and $r > s \geq 0$, namely*

$$\begin{array}{rcl} (n, n-3, 1, n-1, 0) & : & F_n + F_{n-3} = 2F_{n-1} \quad \textit{for} \quad n \geq 3 \quad \textit{and,} \\ (n, n-6, 1, n-2, n-4) & : & F_n + F_{n-6} = 2(F_{n-2} + F_{n-4}) \quad \textit{for} \quad n \geq 6. \end{array}$$

When $n = 4,\ 7$, in the first and second families, we must take $m = 2$ (instead of $m = 1$), respectively. In addition, putting $N := F_n + F_m$, there are exactly 12 values of $N = F_n + F_m$ yielding 21 more sporadic solutions namely:

$$\begin{array}{rcl} 4 & = & F_4 + F_2 = 2^2 F_2; \\ 8 & = & F_6 = 2^2 F_3 = 2^3 F_2; \\ 16 & = & F_7 + F_4 = 2^2(F_4 + F_2) = 2^3 F_3 = 2^4 F_2; \\ 18 & = & F_7 + F_5 = 2(F_6 + F_2); \\ 24 & = & F_8 + F_4 = 2^2(F_5 + F_2) = 2^3 F_4; \\ 36 & = & F_9 + F_3 = 2^2(F_6 + F_2); \\ 56 & = & F_{10} + F_2 = 2^2(F_7 + F_2) = 2^3(F_5 + F_3); \\ 60 & = & F_{10} + F_5 = 2^2(F_7 + F_3); \\ 92 & = & F_{11} + F_4 = 2^2(F_8 + F_3); \\ 144 & = & F_{12} = 2^2(F_9 + F_3) = 2^3(F_7 + F_5) = 2^4(F_6 + F_2); \\ 288 & = & F_{13} + F_{10} = 2^3(F_9 + F_3) = 2^4(F_7 + F_5) = 2^5(F_6 + F_2); \\ 1008 & = & F_{16} + F_8 = 2^4(F_{10} + F_6). \end{array}$$

Our proof uses elementary considerations, linear forms in logarithms and reduction techniques.

4.2 The proof

4.2.1 The cases $a = 0, 1$

Although mentioned in the title of the subsection, we do not have to deal with the case $a = 0$ because in this case $N = M$ and since we work with the Zeckendorf representations of N and M, we conclude that the only situations are the diagonal degenerate ones, namely $(n, m) = (r, s)$. Thus, $a \geq 1$. Assume next that $n > m$. Then

$$2F_n > F_n + F_m = 2^a(F_r + F_s) \geq 2(F_r + F_s) \geq 2F_r,$$

so $n > r$.

We next deal with the case $a = 1$. We have

$$F_n + F_m = 2(F_r + F_s) = 2F_r + 2F_s = F_{r+1} + F_{r-2} + 2F_s.$$

The case $s = 0$ gives $r = n - 1,\ m = r - 2 = n - 3$, which is the first parametric family. If $s \geq 2$, we then get

$$F_n + F_m = F_{r+1} + F_{r-2} + F_{s+1} + F_{s-2}. \tag{4.2}$$

If $s \leq r - 5$, then the right–hand side of (4.2) has a Zeckendorf decomposition of length 4 (if $s > 2$) or 3 (if $s = 2$), and the left–hand side has a Zeckendorf decomposition of length 2 if $m > 0$ or 1 if $m = 0$, a contradiction.

If $s = r - 4$, then the right–hand side of (4.2) is

$$F_{r+1} + (F_{r-2} + F_{r-3}) + F_{r-6} = F_{r+1} + F_{r-1} + F_{r-6}.$$

This is a Zeckendorf decomposition of length 3 except if $r = 6$, when it is a Zeckendorf decomposition with two terms namely $F_7 + F_5$. This gives $(n, m, r, s) = (7, 5, 6, 2)$, which gives the only sporadic solution with $a = 1$ for which $N = 18$.

If $s = r - 3$, then the right–hand side of (4.2) is

$$F_{r+1} + 2F_{r-2} + F_{r-5} = F_{r+1} + F_{r-1} + F_{r-4} + F_{r-5} = F_{r+1} + F_{r-1} + F_{r-3},$$

which is a Zeckendorf decomposition with 3 terms, which is not convenient for us.

Finally, if $s = r - 2$, we then get that the right–hand side of (4.2) is

$$F_{r+1} + (F_{r-1} + F_{r-2}) + F_{r-4} = F_{r+1} + F_r + F_{r-4} = F_{r+2} + F_{r-4},$$

and this is a Zeckendorf decomposition of length 2 if $r > 4$ and of length 1 if $r = 4$. This gives $r = n - 2,\ m = r - 4 = n - 6$ and $s = r - 2 = n - 4$, which is the second parametric family of solutions.

From now on, we may assume that $a \geq 2$.

4.2.2 Bounding a in terms of n and r

Let $(\alpha, \beta) = ((1 + \sqrt{5})/2, (1 - \sqrt{5})/2)$ be the roots of the equation $x^2 - x - 1$.

It is well-known that the Binet formula

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \qquad \text{holds for all} \qquad n \geq 0.$$

We use that

$$\alpha^{k-2} \leq F_k \leq \alpha^{k-1} \qquad \text{for all} \qquad k \geq 1,$$

to get

$$\alpha^{n-2} \leq F_n + F_m = 2^a(F_r + F_s) \leq 2^a(2F_r) \leq 2^{a+1}\alpha^{r-1},$$

so

$$2^{a+1} \geq \alpha^{n-r-1}.$$

Also,

$$2\alpha^{n-1} \geq 2F_n \geq F_n + F_m = 2^a(F_r + F_s) \geq 2^a F_r \geq 2^a \alpha^{r-2},$$

which gives

$$2^{a-1} \leq \alpha^{n-r+1}.$$

Them it follows from the above the following inequalities.

Lemma 4.1. *The inequalities*

$$2^{a-1} \leq \alpha^{n-r+1} \qquad \textit{and} \qquad 2^{a+1} \geq \alpha^{n-r-1}$$

hold.

4.2.3 Six linear forms in logarithms

We take $C_1 := 10^{10},\ C_2 := 10^{12}$,

$$f_i(n) := C_1(2.2C_2)^{i-1}(1+\log n)^i, \qquad i = 1,2,3,4,$$

and put

$$\mathcal{T} = \{n-m, r-s, r+s, n\} = \{t_1, t_2, t_3, t_4\},$$

where $t_1 \le t_2 \le t_3 \le t_4$. We prove the following lemma.

Lemma 4.2. *We have*

$$t_i \le f_i(n), \quad \textit{for} \quad i = 1,2,3,4.$$

Notice that the lemma gives

$$n \le t_4 \le f_4(n), \qquad \text{which gives} \qquad n < 10^{56}. \tag{4.3}$$

In the next section, we will lower the upper bound for n.

Proof. We will apply Matveev's theorem to 6 linear forms in logarithms labelled

$$\Lambda_1, \Lambda_{\{2,1\}}, \Lambda_{\{2,2\}}, \Lambda_{\{3,1\}}, \Lambda_{\{3,2\}}, \Lambda_4,$$

where

$$\Lambda_U := \alpha^{-(n-\delta_U r)}2^a\eta_U - 1, \quad U \in \{1, \{2,1\}, \{2,2\}, \{3,1\}, \{3,2\}, 4\}, \tag{4.4}$$

where $\delta_U = 1$ except for $U \in \{\{3,1\}, 4\}$ when $\delta_U = 0$, and with

$$\begin{aligned} \eta_1 &:= 1, \quad \eta_{2,1} := (1+\alpha^{m-n})^{-1}, \quad \eta_{2,2} := 1+\alpha^{s-r}, \\ \eta_{3,1} &:= \sqrt{5}(F_r+F_s), \quad \eta_{3,2} := \frac{1+\alpha^{r-s}}{1+\alpha^{m-n}}, \quad \eta_4 := \frac{\sqrt{5}(F_r+F_s)}{1+\alpha^{m-n}}. \end{aligned} \tag{4.5}$$

In this chapter when $U = \{a,b\}$, for sake of simplicity, we write $U = a,b$ instead $U = \{a,b\}$.

In order to deduce the upper bounds on t_i, we show that

$$|\Lambda_{U_i}| < \frac{100}{\alpha^{t_{i+1}}}, \qquad \text{for} \qquad i = 0,1,2,3, \tag{4.6}$$

where $U_0 = 1,\ U_1 \in \{\{2,1\},\ \{2,2\}\},\ U_2 \in \{\{3,1\}, \{3,2\}\},\ U_3 = 4$. We also show that $\Lambda_{U_i} \ne 0$ for any $i = 0,1,2,3$, and we show that (4.6) implies, via Matveev's theorem and recursively on i, that $t_{i+1} \le f_{i+1}(n)$.

Since we have many things to prove, we will first explain how to deduce inequalities (4.6) for $i = 0, 1, 2, 3$. Then we will show how inequality (4.6) for $i = 0$ implies $t_1 \le f_1(n)$. Then, for $i \ge 1$, we show how inequality (4.6) for i, Matveev's theorem, the assumption that $\Lambda_{U_i} \ne 0$, and the fact that $t_{j+1} \le f_{j+1}(n)$ holds for $j = 0, 1, \ldots, i-1$, implies that $t_{i+1} \le f_{i+1}(n)$.

So, let us first see how they work. Let $i = 0$. We rewrite our equation (4.1) using the Binet formula for the Fibonacci numbers as

$$\frac{\alpha^n - \beta^n}{\sqrt{5}} + \frac{\alpha^m - \beta^m}{\sqrt{5}} = 2^a \left(\frac{\alpha^r - \beta^r}{\sqrt{5}} + \frac{\alpha^s - \beta^s}{\sqrt{5}} \right),$$

giving

$$\begin{aligned} |\alpha^n - 2^a\alpha^r| &= |\beta^n - \alpha^m + \beta^m + 2^a\alpha^s - 2^a\beta^r - 2^a\beta^s| \\ &\le |\beta|^n + |\beta|^m + \alpha^m + 2^a\alpha^s + 2^a|\beta|^r + 2^a|\beta|^s \\ &\le 2 + \alpha^m + 2^a\alpha^s + 2^{a+1} \le 3(\alpha^m + 2^a\alpha^s) \\ &\le 3(\alpha^m + 2\alpha^{n-r+s+1}) \le 3(2\alpha+1)\alpha^{\max\{m, n-r+s\}}, \end{aligned}$$

where in the above we used that $|\beta| < 1$ and Lemma 4.1. Dividing across by α^n, we get

$$|\Lambda_1| = |\alpha^{-(n-r)}2^a - 1| < \frac{3(2\alpha+1)}{\alpha^{\min\{n-m, r-s\}}} < \frac{100}{\alpha^{t_1}}, \tag{4.7}$$

which we recognise as (4.6) for $i = 0$. Note that we also get that $t_1 = \min\{n-m, r-s\}$. In the same way, we prove that (4.6) holds for $i = 1, 2, 3$. Let's see the details.

For $i = 1$, if $t_1 = n - m$, then (4.1) implies that

$$\begin{aligned} |\alpha^n(1+\alpha^{m-n}) - 2^a\alpha^r| &= |\beta^n + \beta^m + 2^a\alpha^s - 2^a\beta^r - 2^a\beta^s| \\ &\le 2 + 2^a\alpha^s + 2^{a+1} < 3(1 + 2^a\alpha^s) \\ &< 3(1 + 2\alpha^{n-r+s+1}) < 3(2\alpha+1)\alpha^{n-r+s}, \end{aligned}$$

so, dividing across by $\alpha^n(1+\alpha^{m-n})$, we get

$$|\Lambda_{2,1}| = |\alpha^{-(n-r)}2^a(1+\alpha^{m-n})^{-1} - 1| < \frac{3(2\alpha+1)}{\alpha^{r-s}(1+\alpha^{m-n})} < \frac{100}{\alpha^{t_2}}, \tag{4.8}$$

which is (4.6) at $i = 2$. We also note that in this case $t_1 = n - m,\ t_2 = r - s$. On the other hand, if $t_1 = r - s$, then it follow from (4.1) that

$$\begin{aligned} |\alpha^n - 2^a\alpha^r(1+\alpha^{s-r})| &= |-\alpha^m + \beta^n + \beta^m - 2^a\beta^r - 2^a\beta^s| \\ &\le \alpha^m + 2 + 2^{a+1}|\beta|^s = \alpha^m + 2 + 2^{a+1}\alpha^{-s} \\ &< 3(\alpha^m + 2\alpha^{n-r-s+1}) < 3(2\alpha+1)\alpha^{\max\{m, n-r-s\}}, \end{aligned}$$

and dividing across by α^n, we get

$$|\Lambda_{2,2}| = |\alpha^{-(n-r)}2^a(1+\alpha^{s-r}) - 1| < \frac{3(2\alpha+1)}{\alpha^{\min\{n-m,r+s\}}} \leq \frac{100}{\alpha^{t_2}}. \tag{4.9}$$

Here, $t_1 = r - s$ and $t_2 = \min\{n-m, r+s\}$. A similar argument works for $i = 2$ distinguishing the various possibilities for t_1, t_2. In the most asymmetric case $t_1 = r - s, t_2 = r + s$, we rewrite equation (4.1) and we get that

$$|\alpha^n - 2^a\sqrt{5}(F_r + F_s)| = |-\alpha^m + \beta^m + \beta^n| \leq 3\alpha^m,$$

so, dividing across by α^n we get

$$|\Lambda_{3,1}| = |\alpha^{-n}2^a\sqrt{5}(F_r + F_s) - 1| < \frac{3}{\alpha^{n-m}} < \frac{100}{\alpha^{t_3}}, \tag{4.10}$$

which is what we wanted. In the remaining cases, we have $\{t_1, t_2\} = \{n-m, r-s\}$, and then we get from (4.1) that

$$\begin{aligned} |\alpha^n(1+\alpha^{m-n}) - 2^a\alpha^r(1+\alpha^{s-r})| &= |\beta^m + \beta^n + 2^a\beta^r + 2^a\beta^s| \\ &\leq 2|\beta|^m + 2^{a+1}|\beta|^s < 2\alpha^{-m} + 4\alpha^{n-r-s+1} \\ &\leq (2+4\alpha)\alpha^{\max\{-m,n-r-s\}}, \end{aligned}$$

which after dividing it by $\alpha^n(1+\alpha^{m-n})$ we recognise that it leads to

$$|\Lambda_{3,2}| = \left|\alpha^{-(n-r)}2^a\left(\frac{1+\alpha^{s-r}}{1+\alpha^{m-n}}\right) - 1\right| < \frac{2+4\alpha}{\alpha^{\min\{n+m,r+s\}}(1+\alpha^{m-n})} < \frac{100}{\alpha^{t_3}}. \tag{4.11}$$

Here, we take $t_3 = \min\{r+s, n\}$. Cleary, $n > \max\{n-m, r-s\}$ (because $n > r$), so we cannot have $n \in \{t_1, t_2\}$. If $n \neq t_4$, we then get that $n = t_3$, which leads to $t_4 = r + s < 2n$. Thus, by the $i = 2$ step, we would get that $n < f_3(n)$, and later on $t_4 = r + s < 2n < 2f_3(n) < f_4(n)$. So, the inequality for $i = 3$ follows right away from the inequality of $i = 2$. It remains to study the case when $n = t_4$. In this case, (4.1) implies that

$$|\alpha^n(1+\alpha^{m-n}) - 2^a(\sqrt{5}(F_r + F_s))| = |\beta^n + \beta^m| \leq 2,$$

and dividing across by $\alpha^n(1+\alpha^{m-n})$ we get

$$|\Lambda_4| = \left|\alpha^{-n}2^a\left(\frac{\sqrt{5}(F_r + F_s)}{1+\alpha^{m-n}}\right) - 1\right| < \frac{2}{\alpha^n(1+\alpha^{m-n})} < \frac{2}{\alpha^n} < \frac{100}{\alpha^{t_4}}, \tag{4.12}$$

which is inequality (4.6) at $i = 3$.

Having justified inequalities (4.6), let us see how to deduce the upper bounds on t_i. We will prove that $\Lambda_U \neq 0$ later. So far, to get a lower bound on Λ_U, note that

$$\Lambda_U = \alpha_1^{b_1}\alpha_2^{b_2}\alpha_3^{b_3} - 1,$$

where

$$\alpha_1 = \alpha, \quad \alpha_2 = 2, \quad \alpha_3 = \eta_U, \quad b_1 = -(n - \delta_U r), \quad b_2 = a, \quad b_3 = 1.$$

Notice that $\mathbb{K} := \mathbb{Q}(\alpha)$ has degree $D = 2$ and contains $\alpha_1, \alpha_2, \alpha_3$. Next, $b_1 \leq n$. As for b_2, Lemma 4.1 tells us that $2^{a-1} \leq \alpha^{n-r+1}$. If $r \geq 2$, then we get $a < n$ since $\alpha < 2$. On the other hand, if $r = 1$, then $F_n + F_m = 2^a$, which implies that $n \leq 7$ by the main result in [22], and then the inequalities $t_i < f_i(n)$ hold anyway for all $i = 1, 2, 3, 4$. Thus, we may take $B := n > \max\{|b_1|, |b_2|, |b_3|\}$. We take $A_1 := \log\alpha$, $A_2 := 2\log 2$. At $i = 0$, we take $U_0 := 1$, $\eta_{U_0} = \eta_1 = 1$, so we have a linear form in two logarithms only. By Matveev's Theorem 2.6, we get

$$|\Lambda_{U_0}| > \exp(-C_0(1 + \log n)),$$

where

$$C_0 = 1.4 \times 30^5 \times 2^{4.5} 2^2 (1 + \log 2)(\log\alpha)(2\log 2) < 4 \times 10^9.$$

Applying inequality (4.6) at $i = 0$, we get

$$t_1 \log\alpha < \log 100 + C_0(1 + \log n) < 4.1 \times 10^9(1 + \log n),$$

so

$$t_1 < \frac{4.1}{\log\alpha} \times 10^9(1 + \log n) < 10^{10}(1 + \log n) < f_1(n).$$

This is the start. Assume now that $i \geq 2$ and that $t_j \leq f_j(n)$ has been established for $j = 1, \ldots, i-1$. We apply Matveev's Theorem 2.6 to $|\Lambda_{U_{i-1}}|$. We then get that

$$|\Lambda_{U_{i-1}}| > \exp(-C_3(1 + \log n)(2h(\eta_{U_{i-1}}))),$$

where

$$C_3 = 1.4 \times 30^6 \times 3^{4.5} 2^2 (1 + \log 2)(\log\alpha)(2\log 2) < 7 \times 10^{11}.$$

It remains to bound $h(\eta_{U_{i-1}})$. Note that

$$h(\eta_{U_{i-1}}) \leq \begin{cases} t_1(\log\alpha)/2 + \log 2 & i = 2, \\ (r+s)\log\alpha + \log 2 + (\log 5)/2, & i = 3, \quad \text{or} \\ (r-s)(\log\alpha)/2 + (n-m)(\log\alpha)/2 + 2\log 2, & i = 3, \\ (n-m)(\log\alpha)/2 + (r+s)\log\alpha + 2\log 2 + (\log 5)/2, & i = 4. \end{cases} \tag{4.13}$$

Since $2\log 2 + (\log 5)/2 < 3$, it follows from the above that

$$h(\eta_{U_{i-1}}) < \frac{3t_{i-1}\log\alpha}{2} + 3 < \frac{3}{2}\left(f_{i-1}(n)\log\alpha + 2\right).$$

We thus get that

$$t_i \log\alpha < \log 100 + 7 \times 10^{11} \times 3(f_{i-1}(n)\log\alpha + 2)(1 + \log n),$$

which gives

$$\begin{aligned} t_i &< \frac{\log 100}{\log\alpha} + 7 \times 10^{11} \times 3\left(f_{i-1}(n) + \frac{2}{\log\alpha}\right)(1 + \log n) \\ &< 2.2 \times 10^{12} f_{i-1}(n)(1 + \log n) = f_i(n), \end{aligned}$$

which is what we wanted. In the above, we used the fact that

$$\begin{aligned} \frac{\log 100}{\log\alpha} &+ 7 \times 10^{11} \times 3\left(f_{i-1}(n) + \frac{2}{\log\alpha}\right)(1 + \log n) \\ &< (0.71 \times 10^{12} \times 3)\left(f_{i-1}(n) + \frac{2}{\log\alpha}\right)(1 + \log n) \\ &< (2.13 \times 10^{12})(1.01 f_{i-1}(n))(1 + \log n) \\ &< 2.2 \times 10^{12} f_{i-1}(n)(1 + \log n) = f_i(n), \end{aligned}$$

for any $i \geq 2$ and any $n \geq 2$. □

4.2.4 Justifying that $\Lambda_U \neq 0$

For $i = 0$, the form Λ_1 appears in the left–hand side of (4.7). This is zero if and only if $\alpha^{n-r} = 2^a$. This implies $n = r$ and $a = 0$, which is not allowed. For $i = 1$, the form is the one appearing in the left–hand sides of one of (4.8) or (4.9). This gives

$$\alpha^{-(n-r)} 2^a (1 + \alpha^{-t_1})^{\pm 1} = 1.$$

Taking norms and absolute values in $\mathbb{K}$, we get that

$$2^{2a} = |N(1 + \alpha^{-t_1})|^{\pm 1} = |N(\alpha^{t_1} + 1)|^{\pm 1}.$$

The one with negative exponent cannot hold since $1 + \alpha^{t_1}$ is an algebraic integer. The one with positive exponent gives

$$2^{2a} = (\alpha^{t_1} + 1)(\beta^{t_1} + 1) = (\alpha\beta)^{t_1} + 1 + (\alpha^{t_1} + \beta^{t_1}) = L_{t_1} + 1 + (-1)^{t_1}.$$

If t_1 is odd, we get $L_{t_1} = 2^{2a}$. Since 8 never divides L_k for any k, we get $a = 1$ and $t_1 = 3$. If t_1 is even, we get

$$2^{2a} = L_{t_1} + 2 = \begin{cases} 5F_{t_1/2}^2 & \text{if} \quad 2\| t_1, \\ L_{t_1/2}^2 & \text{if} \quad 4 \mid t_1. \end{cases}$$

The first case is impossible since 5 does not divide 2^{2a}. The second case leads to $L_{t_1/2} = 2^a$ with $t_1/2$ being even, which gives again that $a = 1$. However, the case $a = 1$ was treated by elementary arguments using Zeckendorf decompositions in the first section of the proof and we are in the case $a \geq 2$. Thus, Λ_{U_1} is nonzero.

For $i = 3$, if $\eta_{U_2} = \sqrt{5}(F_r + F_s)$, then the form appears in the left–hand side of (4.10). If it is zero, then $-\alpha^m + \beta^m + \beta^n = 0$. If $m = 0$, we get $\beta^n = 0$, which is impossible, while if $m \neq 0$, then $m \geq 2$, so $\alpha^2 \leq \alpha^m = |\beta^m + \beta^n| < 2$, which is a contradiction. If $i = 3$ and $\eta_{U_2} = (1 + \alpha^{s-r})/(1 + \alpha^{m-n})$, then the form appears in the left–hand side of (4.11). If it zero, we get

$$2^a \alpha^{-(n-r)} = \frac{1 + \alpha^{m-n}}{1 + \alpha^{r-s}}.$$

Taking norms we get

$$2^{2a} = \left| N\left(\frac{1 + \alpha^{m-n}}{1 + \alpha^{s-r}}\right)\right| = \frac{|N(\alpha^{n-m} + 1)|}{|N(\alpha^{r-s} + 1)|} = \frac{L_{n-m} + 1 + (-1)^{n-m}}{L_{r-s} + 1 + (-1)^{r-s}}.$$

If $n - m$ is odd, we get $2^{2a} = L_{n-m}/(L_{r-s} + 1 + (-1)^{r-s})$. Since $8 \nmid L_k$ for any $k \geq 1$, we get that $a = 1$, which is not convenient for us. Thus, $n - m$ is even. If $2\|n - m$, we get

$$2^{2a} = \frac{5F_{(n-m)/2}^2}{L_{r-s} + 1 + (-1)^{r-s}}.$$

If $r - s$ is odd, the denominator in the right–hand side above is L_{r-s} a number coprime to 5, so the above equation is impossible since 5 does not divide 2^{2a}. If $4 \mid r - s$, then the denominator in the right–hand side above is $L_{(r-s)/2}^2$, a number coprime to 5, and we get the same contradiction. Finally, it follows that $2\|r - s$, so the equation is

$$2^a = \frac{F_{(n-m)/2}}{F_{(r-s)/2}}.$$

Since $(n - m)/2$ is odd, it follows that $F_{(n-m)/2}$ is even but not a multiple of 4, so $a = 1$, again a contradiction. Finally, if $4 \mid n - m$, we get

$$2^{2a} = \frac{L_{(n-m)/2}^2}{L_{r-s} + 1 + (-1)^{r-s}}.$$

Note that $L_{(n-m)/2}$ can be even but not a multiple of 4 since $(n - m)/2$ is even. This shows again that $a = 1$, a contradiction. Thus, $\Lambda_{U_2} \neq 0$ in all cases. When $i = 4$, the form Λ_{U_3} appears in the left–hand side of (4.12). The condition $\Lambda_{U_3} = 0$ then implies $\beta^n + \beta^m = 0$, so $\beta^{n-m} = -1$, which is impossible. Thus, $\Lambda_{U_3} \neq 0$.

4.2.5 Lowering the bounds

We need to find better bounds on t_i for $i = 1, 2, 3, 4$ then the ones implied by Lemma 4.2 for $n < 10^{56}$.

Bounding t_1

Assume $t_1 \geq 12$. Then the right–hand side of inequality (4.7) is $< 1/2$. It thus follows that

$$|a \log 2 - (n-r)\log\alpha| < \frac{200}{\alpha^{t_1}}.$$

Dividing across by $(n-r)\log 2$, we get

$$\left|\frac{a}{n-r} - \frac{\log\alpha}{\log 2}\right| < \frac{200}{(n-r)(\log 2)\alpha^{t_1}}. \tag{4.14}$$

Suppose that $t_1 \geq 290$. Then the right–hand side above is smaller than $1/(2(n-r)^2)$. Indeed, the inequality

$$\frac{200}{(n-r)(\log 2)\alpha^{t_1}} < \frac{1}{2(n-r)^2},$$

is equivalent to

$$\left(\frac{400}{\log 2}\right)(n-r) < \alpha^{t_1}$$

and this last inequality is fulfilled for $t_1 > 290$ since $n - r \leq n < 10^{56}$. By Legendre's result Lemma 2.1, $a/(n-r) = p_k/q_k$ for some convergent p_k/q_k of $(\log\alpha)/(\log 2)$. Since $q_{113} \leq 10^{56} < q_{114}$, it follows that $k \leq 113$. Since $\max\{a_j : 0 \leq j \leq 114\} = 134$, we get, again by Lemma 2.1, that the left–hand side of (4.14) is bounded below by $1/(136(n-r)^2)$. We thus get that

$$\frac{1}{136(n-r)^2} < \frac{200}{(n-r)(\log 2)\alpha^{t_1}},$$

so

$$\alpha^{t_1} < 136\left(\frac{200}{\log 2}\right)(n-r) < 4 \times 10^{60}, \qquad \text{so} \qquad t_1 < 290,$$

a contradiction. This shows that $t_1 \leq 290$.

Bounding t_2

We assume that $t_2 \geq 300$. We work with inequality (4.8) or (4.9) according to whether $t_1 = n - m$ or $t_1 = r - s$, respectively. In either case, since $100/\alpha^{t_2} < 1/2$, we get that

$$|a \log 2 - (n - r) \log \alpha \pm \log L| < \frac{200}{\alpha^{t_2}} \qquad \text{where} \qquad L := 1 + \alpha^{-t_1}.$$

Dividing both sides by $\log \alpha$, we get

$$|a\tau - (n - r) \pm \mu| < \frac{200}{(\log \alpha)\alpha^{t_2}} < \frac{A}{B^{t_2}}, \tag{4.15}$$

where we take

$$\tau = \frac{\log 2}{\log \alpha}, \quad A = 420, \quad B = \alpha, \quad \mu = \frac{\log(1 + \alpha^{-t_1})}{\log \alpha}, \quad t_1 = 0, 2, \ldots, 290.$$

Note that we did not consider $t_1 = 1$, since t_1 is one of $n - m$ and $r - s$, and we work with Zeckendorf representations of N and M, respectively. In the case $t_1 = 0, 3$, we get

$$\mu = \frac{\log 2}{\log \alpha}, \ \frac{\log 2}{\log \alpha} - 1 \in \{\tau, \tau - 1\}, \qquad \text{respectively},$$

and the argument from the analysis of the bound on t_1 (continued fraction of τ) shows that $t_2 \leq 290$. For $t_1 \in \{2, 4, \ldots, 290\}$, we use the Baker-Davenport reduction method. We choose the convergent $p/q := p_{119}/q_{119}$ given by

$$\frac{575293874524155664430003822457716962182866045634665924176218 2}{3993931203496220640429491278118964138612545968185396080381853}.$$

We choose $M := 10^{56}$, so $6M < 3 \times 10^{60} < q$. Then $M\|q\tau\| < 0.00005$, while

$$\|q\mu\| > 0.0023 \qquad \text{for all} \qquad t_1 \in \{2, 4, \ldots, 290\}.$$

Hence, $\|q\mu\| - M\|q\tau\| > \varepsilon := 0.0005$ for our choices of t_1. We thus get that

$$t_2 \leq \frac{\log(Aq\varepsilon^{-1})}{\log B} < 324.$$

Bounding t_3

Here, we need to increase p/q. We choose $p/q = p_{199}/q_{199}$ and get that $q < 1.3 \times 10^{103}$. We compute $M\|\tau q\| < 1.7 \times 10^{-47}$. In the asymmetric case $t_1 = r - s$, $t_2 = r + s$, we have $2r = t_1 + t_2 < 620$, so $r < 310$. We generated all numbers of the form $\mu := (\log(\sqrt{5}(F_r + F_s)))/(\log \alpha)$ with $0 \le s \le r - 2 < 310$. They appear in the analog of (4.15) with t_2 replaced by t_3 which is

$$|a\tau - (n - \delta_U r) \pm \mu| < \frac{200}{(\log \alpha)\alpha^{t_3}} < \frac{A}{B^{t_3}}. \tag{4.16}$$

In our particular situation, $\delta_U = 0$. Computing $\|q\mu\|$, we get that this number is $> 1.6 \times 10^{-37}$ in all cases. Hence, $1.6 \times 10^{-37} - 1.7 \times 10^{-47} > \varepsilon := 10^{-37}$. We then get that

$$t_3 < \frac{\log(Aq\varepsilon^{-1})}{\log \alpha} < 683.$$

In the case when $\{t_1, t_2\} = \{n - m, r - s\}$, we computed $(1 + \alpha^{-t_2})/(1 + \alpha^{-t_1})$ for $2 \le t_1 < t_2 < 324$. We ignore the case $t_1 = t_2$ since then $\eta_U = 1$ and $t_3 \le 290$ by using the continued fraction of τ as in the bound for t_1. We also ignored the case $\{t_1, t_2\} = \{2, 6\}$. Indeed, if say $n - m = 6, r - s = 2$, we then have $F_n + F_m = F_n + F_{n-6} = 2(F_{n-2} + F_{n-4})$, so we get $2(F_{n-2} + F_{n-4}) = 2^a(F_r + F_{r-2})$. This gives $F_{n-2} + F_{n-4} = 2^{a-1}(F_r + F_{r-2})$. The case $a = 1$ gives the second known parametric family. The case $a - 1 > 0$, yields a new solution $(n', m', a', r', s') = (n - 2, n - 4, a - 1, r, r - 2)$ with $n' - m' = 2 = r' - s'$, so $\eta_U = 1$, showing that $t_3 \le 290$.

The case $r - s = 6, n - m = 2$ is similar, namely we have

$$F_n + F_{n-2} = F_n + F_m = 2^a(F_r + F_{r-6}) = 2^{a+1}(F_{r-2} + F_{r-4}),$$

so we got a new solution $(n', m', a', r', s') = (n, m, a + 1, r - 2, r - 4)$ with $n' - m' = r' - s' = 2$ and we again get $t_3 \le 290$.

So, now we computed all numbers of the form $\|q\mu\|$ for such values of μ obtaining that the minimum exceeds 5.5×10^{-6}. Hence, we can take $\varepsilon := 5 \times 10^{-6}$. We then get

$$t_3 < \frac{\log(Aq\varepsilon^{-1})}{\log \alpha} < 532.$$

To summarise, we have that $t_3 < 683$.

Bounding t_4

There is a lot of work to be done here. First of all, if $n < 683$, we are in good shape. If not $2r = (r + s + r - s) < 683 + 324 < 1100$, so $r < 510$. Having now $s < r < 510$ and

$n - m < 683$, we compute an upper bound on the height of the number $h(\eta_U)$ for $U = 4$ appearing in (4.5). Indeed, by (4.13) we get that $h(\eta_{U_3}) = h(\eta_4) \leq 700$. Using now the upper bound (4.12) on Λ_4 and Matveev's theorem, we obtain

$$n \log \alpha < \log 100 + C_3(700)(1 + \log n) < 5 \times 10^{14}(1 + \log n)$$

giving

$$n < 1.1 \times 10^{15}(1 + \log n),$$

and so $n < 10^{17}$. With this new upper bound for n we go back to the reductions for t_1, t_2, t_3, and repeating the continued fractions arguments and the Baker-Davenport reductions we get $t_1 < 100$, $t_2 < 115$, $t_3 < 235$.

Let us now work on reducing the upper bound for n even more. In fact, if $n < 235$, then we are in good shape. If not $2r = (r + s + r - s) < 235 + 115 < 350$, so $r < 175$. On the other hand, since $2/\alpha^n < 1/2$, from (4.12) we have that

$$|a\tau - n + \nu| < \frac{200}{(\log \alpha)\alpha^{t_4}} < \frac{A}{B^{t_4}}, \tag{4.17}$$

where

$$\nu = \frac{\log(\sqrt{5}(F_r + F_s)/(1 + \alpha^{-(n-m)}))}{\log \alpha}.$$

As mentioned before, Baker-Davenport reduction does not work when μ is a linear combination of 1 and $(\log 2)/(\log \alpha)$ since then $\varepsilon < 0$. In previous cases we identified easily when that was so. That is, when $\mu = (\log(1 + \alpha^{-t_1}))/(\log \alpha)$ the only possibility for $t_1 \geq 2$ for which this number was a linear combination of 1 and $(\log 2)/(\log \alpha)$ was for $t_1 = 3$. Similarly, for $\mu = (\log((1 + \alpha^{-t_2})/(1 + \alpha^{-t_1})))/(\log \alpha)$, the only possibility for $t_3 > t_2 \geq 2$ for which this number was a linear combination of 1 and $(\log 2)/(\log \alpha)$ was for $(t_1, t_2) = (2, 6)$. Here, we have to decide when is the number $\nu = (\log(\sqrt{5}(F_r + F_s)/(1 + \alpha^{-t})))/(\log \alpha)$, where $t = n - m = t_i$ for some $i = 1, 2, 3$ a linear combination of 1 and $(\log 2)/(\log \alpha)$. Well, if this is so, then

$$\frac{\sqrt{5}(F_r + F_s)}{1 + \alpha^{-(n-m)}} = \pm 2^b \alpha^c$$

for some integers b, c. Taking norms in $\mathbb{K}$ and absolute values we get

$$\frac{5(F_r + F_s)^2}{L_{n-m} + 1 + (-1)^{n-m}} = 2^{2b}.$$

If $n - m$ is odd, or $4 \mid n - m$, then the denominator in the left–hand side above is L_{n-m} or $L_{(n-m)/2}^2$. Since $5 \nmid L_k$ for any k, the above equation is impossible. So, $2 \| n - m$, and therefore the denominator in the left–hand side above is $5F_{(n-m)/2}^2$. Hence

$$F_r + F_s = 2^b F_{(n-m)/2}.$$

On the other hand, $2^a(F_r+F_s) = F_n+F_m = F_{(n-m)/2}L_{(n+m)/2}$, where the right–hand side factorisation above holds because $2\|n-m$. Thus, we get $L_{(n+m)/2} = 2^{a+b}$, which implies that $(n+m)/2 = 3$, so $n \le 6$. So, when doing the last Baker-Davenport reduction we eliminate the above instances.

Finally, applying Lemma 2.2 to inequality (4.17), for all choices n, r, s with

$$0 \le s \le r-2 < 173 \quad \text{and} \quad 2 \le n-m \le 235,$$

we obtain that $n \le 400$.

For further convenience of the reader we mention that in the computations above we did not consider the cases $(s, r, n-m) = (0, 2, 2)$ and $(s, r, n-m) = (0, r, 2r)$ with r odd, since then $\varepsilon < 0$ and so Lemma 2.2 does not apply. In fact, if $(s, r, n-m) = (0, 2, 2), (0, r, 2r)$ with r odd, we get that $\nu = 1, r$, respectively. In the first case above we obtain the sporadic solution $F_4+F_2 = 2^2F_2$. In the the second case the original equation is transformed into the simpler equation $L_{m+r} = 2^a$, and so $(m, r, a) = (0, 3, 2)$. Hence we get the solution $F_6 = 2^2F_3$.

4.2.6 The final computation

As we saw in the preceding subsection, it is enough to look for solutions to equation (4.1) for $n \le 400$. What we did, is to generate $F_n + F_m$ for all $m \le n-2 \le 400$. Let $\mathcal{L}_1$ be the set of such numbers. Next, we created a new list $\mathcal{L}_2$ in the following way. For each member N of $\mathcal{L}_1$ for which $4 \mid N$, we put in $\mathcal{L}_2$ the numbers $N/2^k$ for all $k = 2, 3, \ldots, \nu_2(N)$. Here, $\nu_2(N)$ is the exponent of 2 in the factorisation of N. We computed $\mathcal{L}_1 \cap \mathcal{L}_2$ obtaining

$$\mathcal{L}_1 \cap \mathcal{L}_2 = \{1, 2, 3, 4, 6, 7, 9, 14, 15, 18, 23, 36, 63\}.$$

We also found that $\max\{\nu_2(N) : N \in \mathcal{L}_1\} = 18$. From these facts and the original equation (4.1), we can conclude that

$$F_n \le F_n + F_m \le 63 \cdot 2^{18} < 10^8,$$

and therefore $n \le 40$. Then a brute force search with *Mathematica* for $n \le 40$ and $a \ge 2$ gives the sporadic solutions from the statement of the theorem. This completes the proof of Theorem 4.1.

Chapter 5

The 2-adic and 3-adic valuation of the Tripell sequence and an application

Let $\{T_n\}_{n\geq 0}$ denote the Tripell sequence, defined by the third-order linear recurrence $T_n = 2T_{n-1}+T_{n-2}+T_{n-3}$ for $n \geq 3$, with $T_0 = 0$, $T_1 = 1$ and $T_2 = 2$ as initial conditions. In this chapter, we study the 2-adic and 3-adic valuation of the Tripell sequence and, as an application, we determine all Tripell numbers which are factorials.

5.1 Introduction

In number theory, for a given prime number p, the p-adic valuation, or p-adic order, of a non-zero integer n, denoted by $\nu_p(n)$, is the exponent of the highest power of p which divides n. The p-adic order of certain linear recurrence sequences has been studied by many authors. For example, the p-adic order of the Fibonacci numbers was completely characterized by Lengyel in [52]. In 2016, Sanna [70] gave simple formulas for the p-adic order $\nu_p(u_n)$, in terms of $\nu_p(n)$ and the rank of apparition of p in $\{u_n\}_{n\geq 0}$, where $\{u_n\}_{n\geq 0}$ is a nondegenerate Lucas sequence. In particular, from the main theorems of Lengyel and Sanna we extract the following results:

Theorem 5.1. *For each positive integer* n *and each prime number* $p \neq 2, 5$*, we have*

$$\nu_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod 3; \\ 1, & \text{if } n \equiv 3 \pmod 6; \\ 3, & \text{if } n \equiv 6 \pmod{12}; \\ \nu_2(n) + 2, & \text{if } n \equiv 0 \pmod{12}. \end{cases} \tag{5.1}$$

In addition, $\nu_5(F_n) = \nu_5(n)$ *and*

$$\nu_p(F_n) = \begin{cases} \nu_p(n) + \nu_p(F_{\ell(p)}), & \text{if } n \equiv 0 \pmod{\ell(p)}; \\ 0, & \text{otherwise}; \end{cases} \tag{5.2}$$

where $\ell(p)$ *is the least positive integer such that* $p \mid F_{\ell(p)}$.

Theorem 5.2. *For each positive integer* n *and each prime number* $p \neq 2$*, we have that* $\nu_2(P_n) = \nu_2(n)$ *and* $\nu_p(P_n) = \begin{cases} \nu_p(n) + \nu_p(P_{\ell(p)}), & \text{if } n \equiv 0 \pmod{\ell(p)}; \\ 0, & \text{otherwise}; \end{cases}$ *where* $\ell(p)$ *is the least positive integer such that* $p \mid P_{\ell(p)}$.

However, much less is known about the behavior of the p-adic valuation of linear recurrence sequences of higher order. A particular case of linear recurrence sequences of order 3 was studied by Marques and Lengyel in [53]. They characterized the 2-adic valuation of the Tribonacci sequence. The Tribonacci sequence $\{t_n\}_{n\geq 0}$ starts with $t_0 = 0$, $t_1 = 1$, $t_2 = 1$ and satisfies the recurrence $t_n = t_{n-1} + t_{n-2} + t_{n-3}$ for all $n \geq 3$. Results on the 2-adic valuation of Tetra- and Pentanacci numbers can be found in [54]. See also [73, 82] for the behaviour of the 2-adic valuation of generalized Fibonacci numbers and some applications to certain Diophantine equations.

The Pell sequence and its generalizations have been studied by some authors (see [47, 48, 49]). For example, in [48], Kiliç gave some relations involving Fibonacci and generalized Pell numbers showing that generalized Pell numbers can be expressed as the summation of the Fibonacci numbers.

One of the generalizations of the Pell sequence is what we have called the *Tripell sequence* $\{T_n\}_{n\geq 0}$. This sequence starts with $T_0 = 0$, $T_1 = 1$, $T_2 = 2$ and each term afterward is given by the recurrence

$$T_n = 2T_{n-1} + T_{n-2} + T_{n-3}. \tag{5.3}$$

Below we present the first few elements of the Tripell sequence:

$$0, 1, 2, 5, 13, 33, 84, 214, 545, 1388, 3535, 9003, 22929, 58396, \ldots \tag{5.4}$$

In this research we use the theory of constructing identities given by Zhou in [83] and several congruence results to partially characterize the 2-adic valuation of the Tripell sequence and fully characterize the 3-adic valuation $\nu_3(T_n)$.

We next present our theorems in which we give simple formulas for the 2-adic valuation $\nu_2(T_n)$ (for most of the values of n) and the 3-adic valuation $\nu_3(T_n)$ of the Tripell numbers in terms of $\nu_2(n)$ and $\nu_3(n)$, respectively.

Theorem 5.3. *The 2-adic valuation of the nth Tripell number is given by*

$$\nu_2(T_n) = \begin{cases} 0, & \text{if } n \equiv 1,3,4,5 \pmod 7; \\ 2, & \text{if } n \equiv 9 \pmod{14}; \\ 1, & \text{if } n \equiv 2,7 \pmod{14}; \\ \nu_2(n)+1, & \text{if } n \equiv 0 \pmod{14}; \\ \nu_2(n+1)+1, & \text{if } n \equiv 13 \pmod{14}. \end{cases}$$

If $n \equiv 6 \pmod{14}$*, then* $\nu_2(T_n) = \nu_2(n) + 1$ *except when* $n \equiv 1280 \pmod{1792}$ *or, equivalently, when* n *is of the form*

$$n = 14(t2^7 + 2^6 + 2^4 + 2^3 + 2 + 1) + 6 = 1792t + 1280 \quad \textit{with} \quad t \geq 0.$$

Figure 5.1 shows the first few values of $\nu_2(T_n)$.

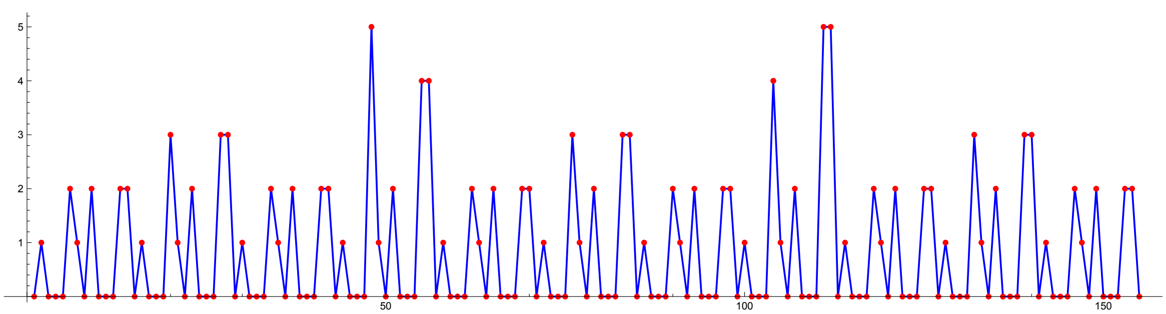

Figure 5.1: The 2-adic valuation of the Tripell numbers

Theorem 5.4. *The 3-adic valuation of the nth Tripell number is given by*

$$\nu_3(T_n) = \begin{cases} 0, & \text{if } n \equiv 1,2,3,4 \pmod 6; \\ \nu_3(n), & \text{if } n \equiv 0 \pmod 6; \\ \nu_3(n+1), & \text{if } n \equiv 5 \pmod 6. \end{cases}$$

As a consequence, we notice that $\nu_3(T_{2n+1}) = \nu_3(T_{2n+2})$ for $n \geq 1$. Figure 5.2 shows the first few values of $\nu_3(T_n)$.

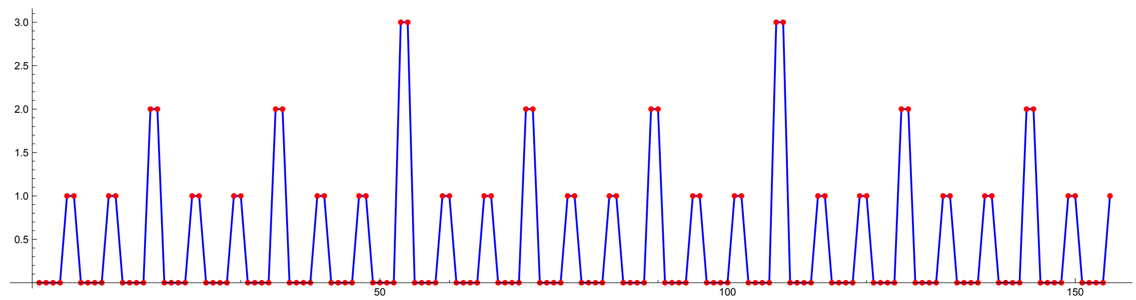

Figure 5.2: The 3-adic valuation of the Tripell numbers

There are many papers in the literature dealing with Diophantine equations obtained by asking that members of some fixed binary recurrence sequence be factorials or belong to some other interesting sequence of positive integers. For example, in [10], all Fibonacci numbers which are sums of three factorials were found, while in [56], all factorials which are sums of three Fibonacci numbers were found.

We also present an application of Theorem 5.4, in which we determine all Tripell numbers which are factorials. We have the following result.

Theorem 5.5. *The only solutions of the Diophantine equation*

$$T_n = m! \tag{5.5}$$

in positive integers n, m are $(n, m) \in \{(1, 1), (2, 2)\}$.

We point out that for finding factorials belonging to some binary recurrence sequences, or related problems, the existence of primitive divisors (see [9]) and other divisibility properties are sometimes used. However, similar divisibility properties for linear recurrences of higher order are not known, and therefore it is necessary to attack the problem differently. It turns out that one can use the p-adic valuation of the terms of these sequences and use it to establish upper bounds on the solutions of some Diophantine equations.

In this chapter we prove Theorem 5.5 by a simple method which makes use of the 3-adic valuation of the terms of the Tripell sequence.

5.2 Auxiliary results

In this section, we present some auxiliary results that are needed in the proofs of the main theorems. To begin with, we give an auxiliary lemma, which is a consequence of Legendre's formula for $\nu_p(m!)$ (see [51]).

Lemma 5.1. *For any integer $m \geq 1$ and prime p, we have*

$$\frac{m}{p-1} - \left\lfloor \frac{\log m}{\log p} \right\rfloor - 1 \leq \nu_p(m!) \leq \frac{m-1}{p-1},$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

A proof of Lemma 5.1 can be found in [58].

We next mention some facts about the Tripell sequence, which will be used later. First, it is easily checked that its characteristic polynomial $f(x) = x^3 - 2x^2 - x - 1$ is irreducible in $\mathbb{Q}[x]$. In addition, $f(x)$ has a real root $\zeta > 1$ and two conjugate complex roots inside the unit circle. In fact,

$$\zeta = \frac{1}{3}\left(2 + \sqrt[3]{\frac{61}{2} - \frac{9\sqrt{29}}{2}} + \sqrt[3]{\frac{61}{2} + \frac{9\sqrt{29}}{2}}\right) = 2.54682\ldots.$$

The following lemma shows the exponential growth of $\{T_n\}_{n\geq 0}$.

Lemma 5.2. *For all $n \geq 1$, we have*

$$\zeta^{n-2} \leq T_n \leq \zeta^{n-1}.$$

Proof. We prove Lemma 5.2 by using strong induction on n. First, note that the result is true for $n = 1, 2, 3$ because

$$\zeta^{-1} \leq T_1 = 1 \leq \zeta^0, \quad \zeta^0 \leq T_2 = 2 \leq \zeta^1, \quad \text{and} \quad \zeta^1 \leq T_3 = 5 \leq \zeta^2.$$

Suppose now that the inequality $\zeta^{m-2} \leq T_m \leq \zeta^{m-1}$ holds for all m with $1 \leq m \leq n-1$. It then follows from the recurrence relation for $(T_n)_{n\geq 0}$ (5.3) that

$$2\zeta^{n-3} + \zeta^{n-4} + \zeta^{n-5} \leq T_n \leq 2\zeta^{n-2} + \zeta^{n-3} + \zeta^{n-4}.$$

So

$$\zeta^{n-5}(2\zeta^2 + \zeta + 1) \leq T_n \leq \zeta^{n-4}(2\zeta^2 + \zeta + 1),$$

which, combined with the fact that $\zeta^3 = 2\zeta^2 + \zeta + 1$, gives the desired result. Thus, Lemma 5.2 holds for all positive integers n. □

Now, we apply the Theory of Constructing Identities to obtain the following identity involving Tripell numbers. This result plays a crucial role in the proofs of Theorems 5.3 and 5.4.

Lemma 5.3. *For all m, n, with $m \geq 3$ and $n \geq 0$, we have that*

$$\begin{aligned} T_{m+n} &= T_{m-1}T_{n+2} + (T_{m-2} + T_{m-3})T_{n+1} + T_{m-2}T_n \\ &= T_{m-1}T_{n+2} + (T_m - 2T_{m-1})T_{n+1} + T_{m-2}T_n. \end{aligned}$$

Proof. It is easily seen that the lemma holds for $m = 3$, so we assume that $m \geq 4$. First, note that $h(x) = x^{m+n} - 2x^{m+n-1} - x^{m+n-2} - x^{m+n-3} \equiv 0 \pmod{f(x)}$, where $f(x)$ is the characteristic polynomial of the sequence $\{T_n\}_{n\geq 0}$. Thus,

$$\begin{aligned} &h(x)(T_1 + T_2x^{-1} + \cdots + T_{m-3}x^{-m+4} + T_{m-2}x^{-m+3}) \\ &\quad= T_1x^{m+n} + T_2x^{m+n-1} + T_3x^{m+n-2} + \cdots + T_{m-3}x^{n+4} + T_{m-2}x^{n+3} \\ &\qquad - 2T_1x^{m+n-1} - 2T_2x^{m+n-2} - 2T_3x^{m+n-3} - \cdots - 2T_{m-3}x^{n+3} - 2T_{m-2}x^{n+2} \\ &\qquad - T_1x^{m+n-2} - T_2x^{m+n-3} - T_3x^{m+n-4} - \cdots - T_{m-3}x^{n+2} - T_{m-2}x^{n+1} \\ &\qquad - T_1x^{m+n-3} - T_2x^{m+n-4} - T_3x^{m+n-5} - \cdots - T_{m-3}x^{n+1} - T_{m-2}x^{n} \\ &\quad= T_1x^{m+n} - (2T_{m-2} + T_{m-3} + T_{m-4})x^{n+2} - (T_{m-2} + T_{m-3})x^{n+1} - T_{m-2}x^n \\ &\quad\equiv 0 \pmod{f(x)}. \end{aligned}$$

Since $T_1 = 1$ and $2T_{m-2} + T_{m-3} + T_{m-4} = T_{m-1}$ by (5.3), we get that

$$x^{m+n} \equiv T_{m-1}x^{n+2} + (T_{m-2} + T_{m-3})x^{n+1} + T_{m-2}x^n \pmod{f(x)}.$$

By Theorem 2.1, we have

$$T_{m+n} = T_{m-1}T_{n+2} + (T_{m-2} + T_{m-3})T_{n+1} + T_{m-2}T_n.$$

□

5.3 Proof of Theorem 5.3

In order to prove Theorem 5.3, we first prove the following lemma.

Lemma 5.4. *For all $s, t \geq 1$, we have*

$$T_{2^t 7s-2} \equiv 1 \pmod{2^{t+2}} \quad \textit{and} \quad T_{2^t 7s-i} \equiv \begin{cases} 2^{t+1} \pmod{2^{t+2}}, & \textit{if } s \equiv 1 \pmod 2; \\ 0 \pmod{2^{t+2}}, & \textit{if } s \equiv 0 \pmod 2; \end{cases} \tag{5.6}$$

for $i = 0, 1$.

Proof. We first need to prove the congruences

$$T_{14s-2} \equiv 1 \pmod 8 \quad \text{and} \quad T_{14s-i} \equiv \begin{cases} 4 \pmod 8, & \text{if } s \equiv 1 \pmod 2; \\ 0 \pmod 8, & \text{if } s \equiv 0 \pmod 2; \end{cases} \tag{5.7}$$

for $i = 0, 1$. Indeed, suppose s is odd, so $s = 2r + 1$ for some integer $r \geq 0$. With *Mathematica* we can easily that $(T_n \mod 8)_{n \geq 0}$ is periodic with period 28. By using this fact we have that

$$\begin{aligned} T_{14(2r+1)-2} &= T_{28r+12} \equiv T_{12} = 22929 \equiv 1 \pmod 8, \\ T_{14(2r+1)-1} &= T_{28r+13} \equiv T_{13} = 58396 \equiv 4 \pmod 8, \\ T_{14(2r+1)} &= T_{28r+14} \equiv T_{14} = 148724 \equiv 4 \pmod 8. \end{aligned}$$

This proves that (5.7) holds when s is odd. A similar argument can be applied in the case where s is even. Thus, (5.7) holds for all $s \geq 1$. Now for a fixed s, we use induction on t to prove the congruences given by (5.6). Note that, by (5.7), (5.6) holds for $t = 1$. Suppose now that congruences (5.6) are true for $t - 1$. Suppose further that s is odd. The case when s is even can be handled in a similar way. Thus,

$$T_{2^{t-1}7s-2} \equiv 1 \pmod{2^{t+1}} \quad \text{and} \quad T_{2^{t-1}7s-i} \equiv 2^t \pmod{2^{t+1}},$$

for $i = 0, 1$, and so

$$T_{2^{t-1}7s-2} = 1 + 2^{t+1}k_1, \quad T_{2^{t-1}7s-1} = 2^t + 2^{t+1}k_2, \quad \text{and} \quad T_{2^{t-1}7s} = 2^t + 2^{t+1}k_3,$$

for some integers k_1, k_2 and k_3. In addition, by using the recurrence relation (5.3) once again, we deduce that

$$T_{2^{t-1}7s-2} + T_{2^{t-1}7s-3} = T_{2^{t-1}7s} - 2T_{2^{t-1}7s-1}.$$

We derive from all this and Lemma 5.3 that

$$\begin{aligned} T_{2^t 7s-2} &= T_{(2^{t-1}7s)+(2^{t-1}7s-2)} \\ &= (2^t + 2^{t+1}k_2)(2^t + 2^{t+1}k_3) + (2^t + 2^{t+1}k_3 - 2(2^t + 2^{t+1}k_2))(2^t + 2^{t+1}k_2) \\ &\quad + (1 + 2^{t+1}k_1)(1 + 2^{t+1}k_1) \\ &\equiv 1 \pmod{2^{t+2}}, \end{aligned}$$

as desired. Similarly, we can prove that

$$T_{2^t 7s-1} = T_{(2^{t-1}7s+1)+(2^{t-1}7s-2)} \equiv 2^{t+1} \pmod{2^{t+2}} \quad \text{and}$$
$$T_{2^t 7s} = T_{(2^{t-1}7s+2)+(2^{t-1}7s-2)} \equiv 2^{t+1} \pmod{2^{t+2}}.$$

This completes the proof of Lemma 5.4. □

Proof of Theorem 5.3

To prove this theorem, we need to work on each case separately.

(a) Let $n \equiv a \pmod 7$ with $a \in \{1, 3, 4, 5\}$. Then it is not difficult to see that $n \equiv 7i+a \pmod{28}$ for some $i \in \{0, 1, 2, 3\}$. Since $(T_n \mod 8)_{n\geq 0}$ is periodic with period 28, it follows that $T_n \equiv T_{7i+a} \pmod 8$. But one can check by hand that $T_{7i+a} \equiv 1, 3, 5$ or $7 \pmod 8$, and so $\nu_2(T_n) = 0$.

(b) If $n \equiv 9 \pmod{14}$, then $n \equiv 9$ or $23 \pmod{28}$. By using the periodicity of $(T_n \mod 8)_{n\geq 0}$ and taking into account that $T_9 \equiv T_{23} \equiv 4 \pmod 8$, we conclude that $\nu_2(T_n) = 2$.

(c) Suppose now that $n \equiv a \pmod{14}$ with $a \in \{2, 7\}$. Then $n \equiv a$ or $14 + a \pmod{28}$. Here we have that $T_2 \equiv T_{16} \equiv T_{21} \equiv 2 \pmod 8$ and $T_7 \equiv 6 \pmod 8$. Thus, $\nu_2(T_n) = 1$.

(d) If $n \equiv 0 \pmod{14}$, then $n = 2^t 7s$ for some $s, t \geq 1$ with s odd. Hence $\nu_2(n) = t$. In addition, by Lemma 5.4 we have that $T_n \equiv 2^{t+1} \pmod{2^{t+2}}$. Thus, $\nu_2(T_n) = t + 1 = \nu_2(n) + 1$.

(e) If $n \equiv 13 \pmod{14}$, then $n = 2^t 7s - 1$ for some $s, t \geq 1$ with s odd. From this $\nu_2(n+1) = t$. Further, by Lemma 5.4 we get $T_n \equiv 2^{t+1} \pmod{2^{t+2}}$. Consequently, $\nu_2(T_n) = t + 1 = \nu_2(n + 1) + 1$.

(f) We finally deal with the special case when $n \equiv 6 \pmod{14}$. Here we have to prove that $\nu_2(T_n) = \nu_2(n) + 1$ except for some special case for n that will be fully characterized. In order to do this, we first write n as $n = 14s + 6$ for some $s \geq 1$. We now distinguish two cases.

Case 1. s **is even**. In this case $n \equiv 6 \pmod{28}$ and so $\nu_2(n) = 1$. In addition, since $(T_n \mod 8)_{n\geq 0}$ is periodic with period 28, we can conclude that $T_n \equiv T_6 = 84 \equiv 4 \pmod 8$, and therefore, $\nu_2(T_n) = 2$. Consequently, $\nu_2(T_n) = \nu_2(n) + 1$.

Case 2. s **is odd**. Here one of the following cases must hold (for some integer $t \geq 0$):

$$\begin{array}{ll} (i)\ s = 2^2t+1, & (v)\ s = 2^4t+2+1, \\ (ii)\ s = 2^3t+2^2+2+1, & (vi)\ s = 2^5t+2^3+2+1, \\ (iii)\ s = 2^7t+2^4+2^3+2+1, & (vii)\ s = 2^7t+2^6+2^4+2^3+2+1. \\ (iv)\ s = 2^6t+2^5+2^4+2^3+2+1, & \end{array} \tag{5.8}$$

We shall work only with the first case, when $s = 2^2t+1$, in order to avoid unnecessary repetitions. The other cases, except the last one, are handled in much the same way. We will prove that $\nu_2(T_{14(2^2t+1)+6}) = \nu_2(14(2^2t+1)+6)+1$ for all $t \geq 0$, by using induction on t. First, note that the base case $t = 0$ follows from $\nu_2(20) = 2$ and $\nu_2(T_{20}) = \nu_2(40585304) = 3$. Suppose now that the result holds true for $t-1$. Then with $m = 14(2^2(t-1)+1)$, we have that $\nu_2(T_{m+6}) = 3$ and

$$\begin{aligned} T_{14(2^2t+1)+6} &= T_{56+14(2^2(t-1)+1)+6} \\ &= T_{55}T_{m+8} + (T_{54}+T_{53})T_{m+7} + T_{54}T_{m+6} \\ &= 2^4k_1T_{m+8} + 2^4k_2T_{m+7} + T_{54}2^3k_3, \end{aligned}$$

for some odd integers k_1, k_2 and k_3. By using this and taking into account that T_{54} is odd, we conclude that $\nu_2(T_{14(2^2t+1)+6}) = 3 = \nu_2(14(2^2t)+20)+1$.

Remark 5.1. *As we saw before, we proved Theorem 5.3 in the special case when* $n \equiv 6 \pmod{14}$ *by using mathematical induction on* t *and this technique worked for almost all of the forms given in* (5.8). *However, the induction argument does not work when* $s = 2^7t+2^6+2^4+2^3+2+1$, *since the base case* $t = 0$ *does not hold.*

5.4 Proof of Theorem 5.4

Here we discuss the 3-adic valuation of the Tripell numbers in a similar way as in the previous section.

Lemma 5.5. *For all* $s, t \geq 1$, $s \not\equiv 0 \pmod 3$, *we have*

$$T_{2\cdot 3^t s-1} \equiv \begin{cases} 2\cdot 3^t \pmod{3^{t+1}}, & \textit{if } s \equiv 1 \pmod 3; \\ 3^t \pmod{3^{t+1}}, & \textit{if } s \equiv 2 \pmod 3; \end{cases}$$

$$T_{2\cdot 3^t s} \equiv \begin{cases} 3^t \pmod{3^{t+1}}, & \textit{if } s \equiv 1 \pmod 3; \\ 2\cdot 3^t \pmod{3^{t+1}}, & \textit{if } s \equiv 2 \pmod 3; \end{cases}$$

and

$$T_{2\cdot 3^t s+1} \equiv \begin{cases} 1+2\cdot 3^t \pmod{3^{t+1}}, & \text{if } s\equiv 1 \pmod 3; \\ 1+3^t \pmod{3^{t+1}}, & \text{if } s\equiv 2 \pmod 3. \end{cases}$$

Proof. The proof proceeds in a similar way as that of Lemma 5.4. Indeed, using the fact that $(T_n \mod 9)_{n\geq 0}$ is periodic with period 18, one can prove that the congruences are valid for all s and $t=1$ (details are left to the reader). Thus, we shall consider the general case for any t and fixed s.

Suppose first that $s\equiv 1 \pmod 3$ and the congruences of the lemma are true for $t-1$. Hence $T_{2\cdot 3^{t-1}s-2} \equiv 1+3^{t-1} \pmod{3^t}$ and consequently

$$\begin{aligned} T_{2\cdot 3^{t-1}s-2} &= 1+3^{t-1}+3^t k_3, & T_{2\cdot 3^{t-1}s-1} &= 2\cdot 3^{t-1}+3^t k_2, \\ T_{2\cdot 3^{t-1}s} &= 3^{t-1}+3^t k_1, & T_{2\cdot 3^{t-1}s+1} &= 1+2\cdot 3^{t-1}+3^t k_4, \end{aligned} \tag{5.9}$$

for some integers k_1, k_2, k_3 and k_4. We next show that congruences of the lemma are also true for t. To do this, we first need to compute $T_{2(2\cdot 3^{t-1}s)+i}$ for $i\in\{-1,0,1\}$. Indeed, by applying the summation identity from Lemma 5.3 and (5.9) we obtain

$$\begin{aligned} T_a := T_{2(2\cdot 3^{t-1}s)-1} &= T_{(2\cdot 3^{t-1}s+1)+(2\cdot 3^{t-1}s-2)} \\ &\equiv 2\cdot 3^{t-1}+3^t k_2+2\cdot 3^{t-1}+3^t k_2 \pmod{3^{t+1}} \\ &\equiv 3^{t-1}+3^t+2\cdot 3^t k_2 \pmod{3^{t+1}}, \\ T_b := T_{2(2\cdot 3^{t-1}s)} &= T_{(2\cdot 3^{t-1}s+2)+(2\cdot 3^{t-1}s-2)} \\ &\equiv 3^{t-1}+3^t k_1+2\cdot 3^{2t-2}+3^{t-1}+3^{2t-2}+3^t k_1 \pmod{3^{t+1}} \\ &\equiv 2\cdot 3^{t-1}+2\cdot 3^t k_1 \pmod{3^{t+1}}, \end{aligned}$$

and

$$\begin{aligned} T_c := T_{2(2\cdot 3^{t-1}s)+1} &= T_{(2\cdot 3^{t-1}s+2)+(2\cdot 3^{t-1}s-1)} \\ &\equiv 1+2\cdot 3^{t-1}+2\cdot 3^{t-1}+3^t k_4+3^t k_4+4\cdot 2^{2t-2}+2\cdot 3^{2t-2} \pmod{3^{t+1}} \\ &\equiv 1+3^{t-1}+3^t+2\cdot 3^t k_4 \pmod{3^{t+1}}. \end{aligned}$$

We thus get that

$$\begin{aligned} T_{2\cdot 3^t s-1} &= T_{(2\cdot 3^{t-1}s)+(2(2\cdot 3^{t-1}s)-1)} \\ &= T_{2\cdot 3^{t-1}s-1}T_c+(T_{2\cdot 3^{t-1}s}-2T_{2\cdot 3^{t-1}s-1})T_b+T_{2\cdot 3^{t-1}s-2}T_a \\ &\equiv 2\cdot 3^{t-1}+3^t k_2+2\cdot 3^{2t-2}+3^{t-1}+3^{2t-2}+3^t+2\cdot 3^t k_2 \pmod{3^{t+1}} \\ &\equiv 2\cdot 3^t \pmod{3^{t+1}}, \end{aligned}$$

and

$$\begin{aligned}
T_{2\cdot 3^t s} &= T_{(2\cdot 3^{t-1}s+1)+(2(2\cdot 3^{t-1}s)-1)} \\
&= T_{2\cdot 3^{t-1}s}T_c + (T_{2\cdot 3^{t-1}s-1} + T_{2\cdot 3^{t-1}s-2})T_b + T_{2\cdot 3^{t-1}s-1}T_a \\
&\equiv 3^{t-1} + 3^t k_1 + 3^{2t-2} + 2\cdot 3^{t-1} + 2\cdot 3^t k_1 + 2\cdot 3^{2t-2} \pmod{3^{t+1}} \\
&\equiv 3^t \pmod{3^{t+1}}.
\end{aligned}$$

A similar argument (which we leave to the reader) shows that

$$T_{2\cdot 3^t s+1} \equiv 1 + 2\cdot 3^t \pmod{3^{t+1}}.$$

We now assume that $s \equiv 2 \pmod 3$. Then $s = 3k+2 = (3k+1)+1$ for some $k \in \mathbb{Z}$. In this case, with $m = 2\cdot 3^t(3k+1)$ and using the previously proved result for the case $s \equiv 1 \pmod 3$, we obtain

$$\begin{aligned}
T_{2\cdot 3^t s-1} &= T_{(2\cdot 3^t)+(2\cdot 3^t(3k+1)-1)} \\
&= T_{2\cdot 3^t-1}T_{m+1} + (T_{2\cdot 3^t} - 2T_{2\cdot 3^t-1})T_m + T_{2\cdot 3^t-2}T_{m-1} \\
&\equiv (2\cdot 3^t)(1+2\cdot 3^t) + (3^t - 2(2\cdot 3^t))3^t + (1+3^t)(2\cdot 3^t) \pmod{3^{t+1}} \\
&\equiv 3^t \pmod{3^{t+1}},
\end{aligned}$$

and

$$\begin{aligned}
T_{2\cdot 3^t s} &= T_{(2\cdot 3^t+1)+(2\cdot 3^t(3k+1)-1)} \\
&= T_{2\cdot 3^t}T_{m+1} + (T_{2\cdot 3^t+1} - 2T_{2\cdot 3^t})T_m + T_{2\cdot 3^t-1}T_{m-1} \\
&\equiv 3^t(1+2\cdot 3^t) + (1+2\cdot 3^t - 2\cdot 3^t)3^t + (2\cdot 3^t)(2\cdot 3^t) \pmod{3^{t+1}} \\
&\equiv 2\cdot 3^t \pmod{3^{t+1}},
\end{aligned}$$

as desired. Similarly, we can prove that

$$T_{2\cdot 3^t s+1} \equiv 1 + 3^t \pmod{3^{t+1}}.$$

□

Proof of Theorem 5.4

Suppose first that $n \equiv a \pmod 6$ with $a \in \{-1, 0\}$. Then n can be written as $n = 2\cdot 3^t s + a$ for $s, t \geq 1$ with $s \not\equiv 0 \pmod 3$. Thus, Lemma 5.5 yields $\nu_3(T_{2\cdot 3^t s+a}) = t$, and then

$$\nu_3(T_n) = \nu_3(T_{2\cdot 3^t s+a}) = t = \nu_3(2\cdot 3^t s) = \nu_3(n-a). \tag{5.10}$$

Suppose now that $n \equiv a \pmod 6$ with $a \in \{1, 2, 3, 4\}$. In this case, by using the fact that $(T_n \mod 3)_{n\geq 0}$ is periodic with period 6, we deduce that $T_n \equiv T_a \pmod 3$. However, one can easily check that $T_a \equiv 1$ or $2 \pmod 3$ for all $a \in \{1, 2, 3, 4\}$, and so $\nu_3(T_n) = 0$.

5.5 Proof of Theorem 5.5

In this last section we apply the 3-adic order of the Tripell sequence to completely solve the Diophantine equation (5.5). Indeed, assume first that equation (5.5) holds. If $m \leq 5$, then the only solutions of (5.5) are those shown in Theorem 5.5, so we may assume that $m \geq 6$. Hence the following inequality holds

$$m! < \left(\frac{m}{2}\right)^m. \tag{5.11}$$

On the other hand, by Theorem 5.4 we get that

$$\nu_3(T_n) = \nu_3(m!) \leq \nu_3(n) + \nu_3(n+1).$$

From this and Lemma 5.1, for $p = 3$, we get

$$\frac{m}{2} - \left\lfloor \frac{\log m}{\log 3} \right\rfloor - 1 \leq \nu_3(m!) \leq 2\max\{\nu_3(n), \nu_3(n+1)\} \leq 2\nu_3(n+\delta),$$

where $\delta = 0, 1$. It then follows that

$$3^{\frac{m}{4} - \frac{\log m}{2\log 3} - \frac{1}{2}} \leq 3^{\nu_3(n+\delta)} \leq n + \delta \leq n + 1,$$

leading to

$$\frac{m}{4} - \frac{\log m}{2\log 3} - \frac{1}{2} \leq \frac{\log(n+1)}{\log 3}. \tag{5.12}$$

Additionally, by Lemma 5.2 and (5.11) we have

$$2.54^{n-2} < T_n = m! < (m/2)^m,$$

and hence $n < 2 + 1.1m\log(m/2)$. Inserting this upper bound on n into (5.12), we conclude that

$$\frac{m}{4} - \frac{\log m}{2\log 3} - \frac{1}{2} < \frac{\log(3 + 1.1m\log(m/2))}{\log 3}. \tag{5.13}$$

This last inequality (5.13) implies that $m < 25$ and therefore, $n < 75$. Finally, a computational search with software *Mathematica* revealed that the only solutions to equation (5.5) are those mentioned in Theorem 5.5. Thus, Theorem 5.5 is proved.

Chapter 6

On a variant of the Brocard–Ramanujan equation and an application

In this chapter, we study the variant of the Brocard–Ramanujan Diophantine equation $m! + 1 = u^2$, where u is a member of a sequence of positive integers. Under some technical conditions on the sequence, we prove that this equation has at most finitely many solutions in positive integers m and u. As an application, we completely solve this equation when u is a Tripell number. The Tripell numbers are defined by the recurrence relation $T_n = 2T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 3$, with $T_0 = 0$, $T_1 = 1$ and $T_2 = 2$ as initial conditions.

6.1 Introduction

Brocard (see [24, 25]), and independently Ramanujan (see [67, 68]), posed the problem of finding all positive integer solutions to the Diophantine equation

$$m! + 1 = n^2. \tag{6.1}$$

This is known as the Brocard–Ramanujan Diophantine equation, and it is still an open problem (see [40]). It is expected that the only solutions are $(m, n) = (4, 5), (5, 11), (7, 71)$. Computations by Berndt and Galway [8] showed that there are no other solution in the range $m < 10^9$. In 1993, Overholt [62] proved that the weak form of Szpiro's conjecture implies that equation (6.1) has only finitely many solutions. The weak form of Szpiro's conjecture is a special case of the ABC conjecture and asserts that there exists a constant s such that if A, B, and C are positive integers satisfying $A+B=C$ with $\gcd(A,B)=1$, then $C \leq N(ABC)^s$, where $N(k)$ is the product of all primes dividing k taken without repetition.

Some variations of equation (6.1) have been considered by various authors and we refer the reader to [29, 30, 31, 55] and references therein for additional information and history. For example, Dabrowski [29] studied the Diophantine equation

$$m! + A = n^2, \tag{6.2}$$

where A is a fixed non-zero integer. He proved that if A is not a square, then equation (6.2) has only finitely many integral solutions. He also proved that if A is a square, then the weak form of Szpiro's conjecture implies the finiteness of solutions for equation (6.2). On the other hand, Luca [55] proved that the ABC-conjecture implies that the Diophantine equation

$$P(x) = n!$$

has finitely many solutions for any $P(x) \in \mathbb{Z}[x]$ of degree > 1.

Variants of (6.1) involving linear recurrences have been also studied. For example, Marques [57] investigated the Fibonacci version of the Brocard-Ramanujan Diophantine equation, namely the equation

$$F_m F_{m+1} \cdots F_{m+k-1} + 1 = F_n^2.$$

Szalay [75] and Pongsriiam [66] worked on another version of the Brocard-Ramanujan problem with Fibonacci, Lucas and balancing numbers, extending the result of Marques [57]. Taşci and Sevgi [76] studied Pell and Pell-Lucas numbers associated with the Brocard-Ramanujan equation, while Pink and Szikszai [65] investigated the Brocard–Ramanujan problem with Lucas and associated Lucas sequences.

In this chapter, we consider the equation

$$m! + 1 = u_n^2, \tag{6.3}$$

in non-negative integers m and n, where $\{u_n\}_{n\geq 0}$ is a sequence of positive integers, which can be seen as a variation of the Brocard-Ramanujan equation. We prove the following theorem in which we specify certain conditions that ensure a finite number of solutions of equation (6.1), and in the proof we can extract an upper bound on the solutions.

In what follows, we use the Landau symbol O with its usual meaning. For two sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ we denote $a_n = O(b_n)$ if there exists a positive constant

K such that $|a_n| \leq K|b_n|$ for all sufficiently large n. In addition, for a given prime number p, the p-adic valuation, or p-adic order, of a non-zero integer n, denoted by $\nu_p(n)$, is the exponent of the highest power of p which divides n.

Theorem 6.1. *Let $\{u_n\}_{n\geq 0}$ be a sequence of positive integers such that*

$$n = O(\log u_n). \tag{6.4}$$

Let p be a prime and assume that

$$\nu_p(u_n + 1) = O(n^{C_1}) \quad \text{and} \quad \nu_p(u_n - 1) = O(n^{C_2}) \tag{6.5}$$

for some constants C_1 and C_2 with $\max\{C_1, C_2\} < 1$. Then, the Diophantine equation (6.3) has only a finite number of solutions in non-negative integers m and n.

Remark 6.1. *Let $a \in \{\pm 2\}$ and consider the sequence $\{u_n\}_{n\geq 0}$ given by $u_0 = 0$, $u_1 = 1$ and the binary linear recurrence $u_n = au_{n-1} - u_{n-2}$ for all $n \geq 2$. Here, we can easily see that $\{u_n\}_{n\geq 0}$ is either equal to $\{n\}_{n\geq 0}$ or to $\{(-1)^{n-1}n\}_{n\geq 0}$, and so in any case we have that $n = O(u_n)$. These examples illustrate situations in which the condition (6.4) of Theorem 6.1 does not hold, although it is usually satisfied for linear recurrence sequences. Examples include sequences having exponential growth. In these latter cases we only need to check whether (6.5) holds for some prime p.*

As mentioned earlier, variants of the Brocard-Ramanujan equation involving binary recurrence sequences have been studied. However, much less is known about the Brocard–Ramanujan problem with linear recurrence sequences of higher order. A particular case of linear recurrence sequences of order 3 was studied by Facó and Marques in [38]. They considered the Brocard–Ramanujan equation

$$m! + 1 = t_n^2,$$

where t_n is the nth Tribonacci number. The Tribonacci numbers are defined by the recurrence $t_n = t_{n-1} + t_{n-2} + t_{n-3}$ for all $n \geq 3$, with $t_0 = 0$ and $t_1 = t_2 = 1$ as initial conditions.

Inspired by these results, we also study an analogue of the problem of Facó and Marques treated in [38] with Tribonacci numbers replaced by Tripell numbers. The Tripell sequence $\{T_n\}_{n\geq 0}$ is one of the generalizations of the Pell sequence. This starts with $T_0 = 0$, $T_1 = 1$, $T_2 = 2$ and each term afterwards is given by the recurrence

$$T_n = 2T_{n-1} + T_{n-2} + T_{n-3}. \tag{6.6}$$

Below we present the first few elements of the Tripell sequence:

$$0,\ 1,\ 2,\ 5,\ 13,\ 33,\ 84,\ 214,\ 545,\ 1388,\ 3535,\ 9003,\ 22929,\ 58396, \ldots \tag{6.7}$$

The Tripell numbers and its generalizations have been studied by some authors (see [19, 47, 48, 49]). For example, in [48], Kiliç gave some relations involving Fibonacci and generalized Pell numbers showing that generalized Pell numbers can be expressed as the summation of the Fibonacci numbers. We have the following theorem.

Theorem 6.2. *The only solution of the Diophantine equation*

$$m! + 1 = T_n^2, \tag{6.8}$$

in non-negative integers m and n, is $(m, n) = (4, 3)$.

The reasoning in Theorem 6.1 provides upper bounds on the solutions of the equation (6.8) if we are able to find some parameters that will be seen in the proof of Theorem 6.1. By determining the 3-adic valuation of the terms $T_n \pm 1$, we will show that it is indeed the case for our sequence $\{T_n\}_{n\geq 0}$.

6.2 Auxiliary results

In this section, we present some auxiliary results that are needed in the proofs of the main theorems. To begin with, we give an auxiliary lemma,

Lemma 6.1. *For any integer $m \geq 1$ and prime p, we have*

$$\frac{m}{p-1} - \left\lfloor \frac{\log m}{\log p} \right\rfloor - 1 \leq \nu_p(m!) \leq \frac{m-1}{p-1}, \tag{6.9}$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

We next mention some facts about the Tripell sequence which will be used later. First, it is known that its characteristic polynomial $f(x) = x^3 - 2x^2 - x - 1$ is irreducible in $\mathbb{Q}[x]$. In addition, $f(x)$ has a real root $\zeta > 1$ and two conjugate complex roots inside the unit circle. In fact,

$$\zeta = \frac{1}{3}\left(2 + \sqrt[3]{\frac{61}{2} - \frac{9\sqrt{29}}{2}} + \sqrt[3]{\frac{61}{2} + \frac{9\sqrt{29}}{2}}\right) = 2.54682\ldots. \tag{6.10}$$

The following inequality, which shows the exponential growth of $\{T_n\}_{n=0}^{\infty}$,

Lemma 6.2. *For all* $n \geq 1$*, we have*

$$\zeta^{n-2} \leq T_n \leq \zeta^{n-1}. \tag{6.11}$$

This result plays a crucial role in the proof of Theorem 6.2.

Lemma 6.3. *For all* m, n*, with* $m \geq 3$ *and* $n \geq 0$*, we have that*

$$\begin{aligned} T_{m+n} &= T_{m-1}T_{n+2} + (T_{m-2} + T_{m-3})T_{n+1} + T_{m-2}T_n \\ &= T_{m-1}T_{n+2} + (T_m - 2T_{m-1})T_{n+1} + T_{m-2}T_n. \end{aligned}$$

We finish this section of auxiliary results by giving two congruence lemmas.

Lemma 6.4. *For all* $s, t \geq 1$*,* $s \not\equiv 0 \pmod 3$*, we have*

$$T_{2\cdot 3^t s-1} \equiv \begin{cases} 2\cdot 3^t \pmod{3^{t+1}}, & \textit{if } s \equiv 1 \pmod 3; \\ 3^t \pmod{3^{t+1}}, & \textit{if } s \equiv 2 \pmod 3; \end{cases}$$

$$T_{2\cdot 3^t s} \equiv \begin{cases} 3^t \pmod{3^{t+1}}, & \textit{if } s \equiv 1 \pmod 3; \\ 2\cdot 3^t \pmod{3^{t+1}}, & \textit{if } s \equiv 2 \pmod 3; \end{cases}$$

and

$$T_{2\cdot 3^t s+1} \equiv \begin{cases} 1+2\cdot 3^t \pmod{3^{t+1}}, & \textit{if } s \equiv 1 \pmod 3; \\ 1+3^t \pmod{3^{t+1}}, & \textit{if } s \equiv 2 \pmod 3. \end{cases}$$

As an immediate consequence of the above lemma we have the following congruence. This will be needed for dealing some particular case mentioned in the last part of Lemma 6.6.

Lemma 6.5. *For all* $s, t \geq 1$*,* $s \not\equiv 0 \pmod 3$*, we have*

$$T_{2\cdot 3^t s-4} \equiv \begin{cases} -1+3^t \pmod{3^{t+1}}, & \textit{if } s \equiv 1 \pmod 3; \\ -1+2\cdot 3^t \pmod{3^{t+1}}, & \textit{if } s \equiv 2 \pmod 3. \end{cases}$$

Proof. This result is a direct consequence of the recursive formula

$$T_{n-4} = -T_{n+1} + T_n + 4T_{n-1} \quad \text{for} \quad n \geq 4,$$

which can be obtained by using the recurrence relation for $(T_n)_{n=0}^{\infty}$, and Lemma 6.4. $\square$

6.3 Proof of Theorem 6.1

Suppose that (m, n) is an integer solution of equation (6.3) and the assumptions (6.4) and (6.5) are satisfied for some prime number p. First of all, it is clear that there exists a positive integer m_0 such that the inequality

$$\lfloor \log_p m \rfloor + 1 \le \frac{m}{p(p-1)} = \frac{m}{p-1} - \frac{m}{p}, \tag{6.12}$$

holds for all $m \ge m_0$. From this and Lemma 6.1, we get that

$$\frac{m}{p} \le \frac{m}{p-1} - \lfloor \log_p m \rfloor - 1 \le \nu_p(m!) \tag{6.13}$$

for all $m \ge m_0$. On the other hand, by the assumption (6.5), there exist some positive constants K_1, K_2 and an integer $n_1 \ge 0$ such that

$$\begin{aligned} \frac{m}{p} &\le \nu_p(m!) = \nu_p(u_n^2 - 1) = \nu_p(u_n + 1) + \nu_p(u_n - 1) \\ &\le K_1 n^{C_1} + K_2 n^{C_2} \\ &\le 2Kn^C, \end{aligned}$$

for all $n \ge n_1$, where $K = \max\{K_1, K_2\}$ and $C = \max\{C_1, C_2\}$. Thus,

$$n \ge \left(\frac{m}{2Kp}\right)^{1/C} \tag{6.14}$$

for all $n \ge n_1$ and $m \ge m_0$. By the assumption (6.4), the inequality $n \le K_3 \log u_n$ holds for all $n \ge n_2$, for some positive constant K_3 and some integer $n_2 \ge 0$. Note that there is only a finite number of solutions of equation (6.3) with $m \le 5$. If $m \ge 6$, the inequality $m! + 1 < (m/2)^m$ holds by Stirling's formula, and hence $u_n^2 = m! + 1 < (m/2)^m$ for all $m \ge 6$. Therefore,

$$\frac{2n}{K_3} \le 2 \log u_n < m \log(m/2) \tag{6.15}$$

for all $n \ge n_2$ and $m \ge 6$. Combining (6.14) and (6.15) we obtain that

$$\log(m/2) > \frac{2n}{K_3 m} \ge \hat{K} m^{1/C - 1} \tag{6.16}$$

for all $n \ge \max\{n_1, n_2\}$ and $m \ge \max\{m_0, 6\}$, where $\hat{K} = 2/(K_3(2Kp)^{1/C})$. But $C < 1$ implies that $1/C - 1 > 0$, and so (6.16) holds only for finitely many m. Thus, by inequality (6.15) we obtain an upper bound for n. Finally, we observe that if $n < \max\{n_1, n_2\}$ or $m < \max\{m_0, 6\}$, then (6.3) and (6.4) imply that n and m have finitely many possibilities. Consequently, equation (6.3) has only a finite number of solutions. Theorem 6.1 is therefore proved.

6.4 Proof of Theorem 6.2

We first present the following key result in which we give simple formulas for the 3-adic valuations $\nu_3(T_n \pm 1)$, and will play an important role in the proof of Theorem 6.2.

Lemma 6.6. *For each positive integer n, we have*

$$\nu_3(T_n - 1) = \begin{cases} 0, & \text{if } n \equiv 0, 2, 3, 5 \pmod 6; \\ \nu_3(n-1), & \text{if } n \equiv 1 \pmod 6; \\ \nu_3(n+2), & \text{if } n \equiv 4 \pmod 6; \end{cases} \tag{6.17}$$

and

$$\nu_3(T_n + 1) = \begin{cases} 0, & \text{if } n \equiv 0, 1, 4, 5 \pmod 6; \\ \nu_3(n+4), & \text{if } n \equiv 2 \pmod 6; \\ \nu_3(n+3), & \text{if } n \equiv 3 \pmod 6. \end{cases} \tag{6.18}$$

Proof of Lemma 6.6

(*a*) Let $n \equiv a \pmod 6$ with $a \in \{0, -1\}$. Then $n = 2 \cdot 3^t s + a$ for some $t \geq 1$ and $s \not\equiv 0 \pmod 3$. Here, it follows directly from Lemma 6.4 that $\nu_3(T_n \pm 1) = 0$.

(*b*) If $n \equiv 1 \pmod 6$, then $n = 2 \cdot 3^t s + 1$ for some $t \geq 1$ and $s \not\equiv 0 \pmod 3$. By Lemma 6.4 we get that $\nu_3(T_n - 1) = t = \nu_3(n-1)$ and $\nu_3(T_n + 1) = 0$.

(*c*) Suppose now that $n \equiv 3 \equiv -3 \pmod 6$. Then $n = 2 \cdot 3^t s - 3$ for some $t \geq 1$ and $s \not\equiv 0 \pmod 3$. It is a simple matter to show, by using the recurrence relation (6.6), that $T_n = 3T_{n+3} - T_{n+2} - T_{n+4}$. From this and Lemma 6.4, we have

$$T_n \equiv \begin{cases} -1 - 3^t \pmod{3^{t+1}}, & \text{if } s \equiv 1 \pmod 3; \\ -1 - 2 \cdot 3^t \pmod{3^{t+1}}, & \text{if } s \equiv 2 \pmod 3. \end{cases}$$

Thus, $\nu_3(T_n - 1) = 0$ and $\nu_3(T_n + 1) = t = \nu_3(n+3)$.

(*d*) If $n \equiv 4 \equiv -2 \pmod 6$, then $n = 2 \cdot 3^t s - 2$ for some $t \geq 1$ and $s \not\equiv 0 \pmod 3$. We now use the recursive formula $T_n = T_{n+3} - 2T_{n+2} - T_{n+1}$ and Lemma 6.4 to obtain

$$T_n \equiv \begin{cases} 1 - 2 \cdot 3^t \pmod{3^{t+1}}, & \text{if } s \equiv 1 \pmod 3; \\ 1 - 3^t \pmod{3^{t+1}}, & \text{if } s \equiv 2 \pmod 3. \end{cases}$$

Thus, $\nu_3(T_n - 1) = t = \nu_3(n+2)$ and $\nu_3(T_n + 1) = 0$, as we wanted.

(e) If $n \equiv 2 \pmod 6$, then $n = 2 \cdot 3^t s + 2$ for some $t \geq 1$ and $s \not\equiv 0 \pmod 3$. Here, by using the recurrence (6.6) and Lemma 6.4 once again, we obtain that

$$T_n \equiv \begin{cases} 2 + 3^t \pmod{3^{t+1}}, & \text{if } s \equiv 1 \pmod 3; \\ 2 + 2 \cdot 3^t \pmod{3^{t+1}}, & \text{if } s \equiv 2 \pmod 3. \end{cases}$$

Thus, $\nu_3(T_n - 1) = 0$. We also deduce that $\nu_3(T_n + 1) = 1 = \nu_3(n+4)$ except when $t = 1$ and $s \equiv 2 \pmod 3$. We finally deal with the special case when $n = 6s + 2$ with $s \equiv 2 \pmod 3$. Note that, in this special case for n, we can write $n = 2 \cdot 3^r k - 4$ for some $r \geq 2$ and $k \not\equiv 0 \pmod 3$. We thus obtain that $\nu_3(T_n + 1) = r = \nu_3(n+4)$ by applying Lemma 6.5.

6.5 Proof of Theorem 6.2

First of all, by the left-hand side of the inequality of Lemma 6.2 we have that

$$n \log \zeta \leq 2 \log \zeta + \log T_n \leq 1.5 \log T_n \quad \text{for all} \quad n \geq 6,$$

leading to

$$n \leq \frac{1.5}{\log \zeta} \log T_n < 2 \log T_n \quad \text{for all} \quad n \geq 6.$$

Thus, $n = O(\log T_n)$. On the other hand, for $a \in \{-1, 2, 3, 4\}$, we get

$$n + a = 3^{\nu_3(n+a)} x$$

for some integer $x \geq 1$, and therefore,

$$\nu_3(n+a) = \log_3(n+a) - \log_3 x \leq \log_3(n+4) < 1.5 n^{1/3}$$

for all $n \geq 1$. From the above and Lemma 6.6 we conclude that the inequalities

$$\nu_3(T_n - 1) \leq \nu_3(n-1) + \nu_3(n+2) < 3n^{1/3}$$

and

$$\nu_3(T_n + 1) \leq \nu_3(n+3) + \nu_3(n+4) < 3n^{1/3}$$

hold for all $n \geq 1$. Thus, $\nu_3(T_n - 1) = O(n^{1/3})$ and $\nu_3(T_n + 1) = O(n^{1/3})$, and so the assumptions (6.4) and (6.5) of Theorem 6.1 are fulfilled. In the proof of Theorem 6.1 we can take the parameters $(p, m_0, K, K_3, C) = (3, 18, 3, 2, 1/3)$ and so

$$\hat{K} = 2/(K_3(2Kp)^{1/C}) = 0.000171468\cdots.$$

By using this, from (6.16) we obtain the inequality

$$m^2 < 5850 \log(m/2)$$

implying that $m \leq 160$. Hence, by (6.15), $n \leq 710$. Finally, a computational search revealed that the only solution to equation (6.8) is $(m, n) = (4, 3)$. Thus, Theorem 6.2 is proved.

Bibliography

[1] M. A. Alekseyev, *On the intersections of Fibonacci, Pell and Lucas numbers*, Integers, **11** (2011), 239–259.

[2] A. Baker, *Linear forms in the logarithms of algebraic numbers I*, Mathematika, **12** (1966), 204–216.

[3] A. Baker, *Linear forms in the logarithms of algebraic numbers II*, Mathematika, **14** (1967), 102–107.

[4] A. Baker, *Linear forms in the logarithms of algebraic numbers III,* Mathematika, **14** (1967), 220–228.

[5] A. Baker, *Linear forms in the logarithms of algebraic numbers IV,* Mathematika, **15** (1968), 204–216.

[6] M. A. Bennett, *On some exponential equations of S. S. Pillai*, Canad. J. Math., **53** (2001), 897-922.

[7] D. Berend and C. Osgood, *On the equation $P(x) = n!$ and a question of Erdos,* J. Number Theory, **42** (1992), 189–193.

[8] B. C. Berndt and W. F. Galway, *On the Brocard-Ramanujan Diophantine equation* $n! + 1 = m^2$, Ramanujan J., **4** (2000), 41–42.

[9] Y. Bilu, G. Hanrot and P. Voutier, *Existence of primitive divisors of Lucas and Lehmer numbers*, J. Reine Angew. Math., **539** (2001), 75–122.

[10] M. Bollman, S. Hernández and F. Luca, *Fibonacci numbers which are sums of three factorials,* Publ. Math. Debrecen, **77** (2010), 211–224.

[11] J. J. Bravo, M. Díaz and C. A. Gómez, *Pillai's problem with k-Fibonacci and Pell numbers*, J. Difference Equ. Appl., (2021), **27** (2021), no. 10, 1434-1455.

[12] J. J. Bravo, M. Díaz and F. Luca, *Ratios of sums of two Fibonacci numbers equal to powers of 2*, Math. Commun., **25** (2020), no. 2, 185–199.

[13] J. J. Bravo, M. Díaz and J. L. Ramírez, *The 2-adic and 3-adic valuation of the Tripell sequence and an application*, Turkish J. Math., **44** (2020), no. 1, 131–141.

[14] J. J. Bravo, M. Díaz and J. L. Ramírez, *On a variant of the Brocard–Ramanujan equation and an application*, Publ. Math. Debrecen, **98** (2021), no. 1, 243–253.

[15] J. J. Bravo, C. A. Gómez and J. L. Herrera, *On the intersection of k-Fibonacci and Pell numbers*, Bull. Korean Math. Soc., **56** (2019), no. 2, 535–547.

[16] J. J. Bravo, C. A. Gómez and F. Luca, *On the distance between generalized Fibonacci numbers*, Colloq. Math., **140** (2015), no. 1, 107–118.

[17] J. J. Bravo, C. A. Gómez and F. Luca, *Powers of two as sums of two k-Fibonacci numbers*, Miskolc Math. Notes, **17** (2016), no. 1, 85-100.

[18] J. J. Bravo, C. A. Gómez and F. Luca, *A Diophantine equation in k-Fibonacci numbers and repdigits*, Colloq. Math., **152** (2018), 299–315.

[19] J. J. Bravo, J. L. Herrera and J. L. Ramírez, *Combinatorial interpretation of generalized Pell numbers*, J. Integer Seq., **23**, (2020), Article 20.2.1.

[20] J. J. Bravo and F. Luca, *Powers of two in generalized Fibonacci sequences*, Rev. Colombiana Mat., **46** (2012), no. 1, 67–79.

[21] J. J. Bravo and and F. Luca, *On a conjecture about repdigits in k-generalized Fibonacci sequences*, Publ. Math. Debrecen, **82** (2013), 623–639.

[22] J. J. Bravo and F. Luca, *On the Diophantine equation $F_n + F_m = 2^a$*, Quaest. Math., **39** (2016), 391–400.

[23] J. J. Bravo, F. Luca and K. Yazán, *On a problem of Pillai with Tribonacci numbers and powers of 2*, Bull. Korean Math. Soc., **54** (2017), no. 3, 1069–1080.

[24] H. Brocard, *Question 166*, Nouv. Corresp. Math., **2** (1876), 287.

[25] H. Brocard, *Question 1532*, Nouv. Ann. Math., **4** (1885), 391.

[26] R. D. Carmichael, *On the Numerical Factors of the Arithmetic Forms $\alpha^n \pm \beta^n$*, Ann. of Math., **15** (1913), no. 1, 30-70.

[27] K. C. Chim, I. Pink, and V. Ziegler, *On a variant of Pillai's problem*, Int. J. Number Theory, **13** (2017), no. 7, 1711-1727.

[28] K. C. Chim, I. Pink, and V. Ziegler, *On a variant of Pillai's problem II*, J. Number Theory, **183** (2018), 269-290.

[29] A. Dabrowski, *On the Diophantine equation* $x! + A = y^2$, Nieuw Arch. Wisk., **14** (1996), 321–324.

[30] A. Dabrowski, *On the Brocard-Ramanujan problem and generalizations*, Colloq. Math., **126** (2012), no. 1, 105–110.

[31] A. Dabrowski and M. Ulas, *Variations on the Brocard-Ramanujan equation*, J. Number Theory, **133** (2013), no. 4, 1168–1185.

[32] M. Ddamulira, C. A. Gómez and F. Luca, *On a problem of Pillai with k-generalized Fibonacci numbers and powers of 2*, Monatsh Math., **187** (2018), 635–664.

[33] M. Ddamulira, F. Luca and M. Rakotomalala, *On a problem of Pillai with Fibonacci numbers and powers of 2*, Proc. Indian Acad. Sci. Math. Sci., **127** (2017), no. 3, 411–421.

[34] G. P. Dresden and Z. Du, *A simplified Binet formula for k-generalized Fibonacci numbers*, J. Integer Sequences, **17** (2014), Article 14.4.7.

[35] A. Dujella and A. Pethő, *A generalization of a theorem of Baker and Davenport*, Quart. J. Math. Oxford Ser., **49** (1998),no. 2, 291–306.

[36] F. J. Dyson, *The approximation to algebraic numbers by rationals*, Acta Math.,**79** (1947), 225-240.

[37] J. Evertse, *Linear forms in logarithms,* Available at http://www.math.leidenuniv.nl/ evertse/dio2011-linforms.pdf, April (2011).

[38] V. Facó and D. Marques, *Tribonacci numbers and the Brocard-Ramanujan equation*, J. Integer Seq., **19** (2016), Article 16.4.4.

[39] A. C. García and S. Hernández, *Pillai's problem with Padovan numbers and powers of two*, Rev. Colombiana Mat., **53** (2019), no. 1, 1–14.

[40] R. K. Guy, *Unsolved Problems in Number Theory*, Springer–Verlag, (1994).

[41] S. Guzmán and F. Luca, *Linear combinations of factorials and S-units in a binary recurrence sequence,* Ann. Math. Qébec, **38** (2014), 169-188.

[42] M. O. Hernane, F. Luca, S. E. Rihane and A. Togbé, *On Pillai's problem with Pell numbers and powers of 2*, Hardy-Ramanujan J., **41** (2018), 22–31.

[43] S. Hernández, F. Luca and L. M. Rivera, *On Pillai's problem with the Fibonacci and Pell sequences*, Bol. Soc. Mat. Mex., **25** (2019), no. 3, 495-507.

[44] A. Herschfeld, *The equation* $2^x - 3^y = d$, Bull. Amer. Math. Soc., **41** (1935), 631.

[45] A. Herschfeld, *The equation* $2^x - 3^y = d$, Bull. Amer. Math. Soc., **42** (1936), 231–234.

[46] F. T. Howard and F. Saidak, *Zhou's theory of constructing of identities* , Congr. Numer., **200** (2010), 225-237.

[47] E. Kiliç, *The Binet formula, sums and representations of generalized Fibonacci p-numbers*, European J. Combin., **29** (2008), no. 3, 701–711.

[48] E. Kiliç, *On the usual Fibonacci and generalized order-k Pell numbers*, Ars Combinatoria, **88** (2008), 33–45.

[49] E. Kiliç and D. Taşci, *The generalized binet formula, representation and sums of the generalized order-k Pell numbers*, Taiwanese J. Math., **10** (2006),no. 6, 1661–1670.

[50] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley–Interscience Publications, 2001.

[51] AM. Legendre, *Théorie des nombres,* Firmin Didot Frères, (1830) (in French).

[52] T. Lengyel *The order of the Fibonacci and Lucas numbers,* Fibonacci Quart., **33** (1995), no. 3, 234–239.

[53] T. Lengyel and D. Marques, *The 2-adic order of Tribonacci numbers and the equation* $T_n = m!$, J. Integer Seq., **17** (2014), Article 14.10.1.

[54] T. Lengyel and D. Marques, *The 2-adic order of some generalized Fibonacci numbers,* Integers, **17** (2017), Article #A5, 10 pp.

[55] F. Luca, *The Diophantine equation* $P(x) = n!$ *and a result of M. Overholt*, Glas. Math. Ser. III, **37** (2002), no. 2, 269–273.

[56] F. Luca and S. Siksek, *Factorials expressible as sums of two and three Fibonacci numbers,* Proc. Edinb. Math. Soc., **53** (2010), 747–763.

[57] D. Marques, *The Fibonacci version of the Brocard-Ramanujan diophantine equation*, Port. Math., **68** (2011), 185–189.

[58] D. Marques, *The order of appearance of product of consecutive Fibonacci numbers,* Fibonacci Quart., **50** (2012), 132–139.

[59] E. M. Matveev, *An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers, II*, Izv. Ross. Akad. Nauk Ser. Mat., **64** (2000), 125–180. English translation in Izv. Math., **64** (2000), 1217–1269.

[60] P. Mihăilescu, *Primary cyclotomic units and a proof of Catalan s conjecture,* J. Reine Angew. Math., **572** (2004), 167-195.

[61] M. R. Murty and J. Esmonde, *Problems in Algebraic Number Theory*, Second Edition, Graduate Texts in Mathematics, Springer-Verlag, New York, 2005.

[62] M. Overholt, *The Diophantine equation* $n! + 1 = m^2$, Bull. Lond. Math. Soc., **25** (1993), 104.

[63] S. S. Pillai. *On* $a^x - b^y = c.$, J. Indian Math. Soc. (N.S.), **2** (1936), 119-122.

[64] S. S. Pillai, *A correction to the paper On* $a^x - b^y = c$, J. Indian Math. Soc., **2** (1937), 215.

[65] I. Pink and M. Szikszai, *A Brocard-Ramanujan-type equation with Lucas and associated Lucas sequences*, Glas. Mat. Ser., **52** (2017), no. 1, 11–21.

[66] P. Pongsriiam, *Fibonacci and Lucas numbers associated with Brocard-Ramanujan equation*, Commun. Korean Math. Soc., **32** (2017), no. 3, 511–522.

[67] S. Ramanujan, *Question 469*, J. Indian Math. Soc., **5** (1913), 59.

[68] S. Ramanujan, *Collected papers*, New York, 1962.

[69] K. F. Roth, *Rational approximations to algebraic numbers*, Mathematika, **2** (1955), 1–20.

[70] C. Sanna, *The p-adic valuation of Lucas sequences,* Fibonacci Quart., **54** (2016), no. 2, 118–124.

[71] C. L. Siegel, *Approximation algebraischer Zahlen*, Math. Zeitschrift, **10** (1921), 173-213.

[72] C. L. Siegel, *Ueber Näherungswerte algebraischer Zahlen,* Math. Ann., **84** (1921), 80-99.

[73] B. Sobolewski, *The 2-adic valuation of generalized Fibonacci sequences with an application to certain Diophantine equations,* J. Number Theory, **180** (2017), 730–742.

[74] R. J. Stroeker and R. Tijdeman, *Diophantine equations*, Mathematisch Centrum Computational Methods in Number Theory, Part II, **155** (1982), 321–369.

[75] L. Szalay, *Diophantine equations with binary recurrences associated to Brocard-Ramanujan problem*, Port. Math., **69** (2012), no. 3, 213–220.

[76] D. Taşci and E. Sevgi, *Pell and Pell-Lucas numbers associated with Brocard-Ramanujan equation*, Turk. J. Math. Comput. Sci., **7** (2017), 59–62.

[77] A. Thue, *Om en generel i store hele tal ulϕsbar ligning, Christiania Videnskabs Selskabets Skrifter*, **7** (1908), 1-15.

[78] A. Thue, *Über Annäherungswerte algebraischer Zahlen* , J. Reine angew. Math., **135** (1909), 284-305.

[79] M. Waldshmidt, *Diophantine Approximation on Linear Algebraic Groups: Transcendence Properties of the Exponential Function in Several Variables*, Springer–Verlag Berlin Heidel-berg, Berlin, 2000.

[80] M. Waldschmidt, *Perfect Powers: Pillai's works and their developments*, arXiv preprint arXiv:0908.4031, (2009).

[81] D. A. Wolfram, *Solving Generalized Fibonacci Recurrences,* Fibonacci Quart., **36** (1998), no. 2, 129-145.

[82] PT. Young, *2-adic valuations of generalized Fibonacci numbers of odd order,* Integers, **18** (2018), Article #A1, 13 pp.

[83] C. Zhou, *Constructing identities involving kth order F-L numbers by using the characteristic polynomial*, Applications of Fibonacci Numbers, **8**, Edited F. T. Howard, (1999), 369–379.